MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

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Solved Questions.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

1878. (Proposed by W. K. CLIFFORD, B.A.)—If a line of given length be marked at random in n points, and broken up at those points, find the chance that the sum of the squares on the parts shall not exceed one-nth of the square on the whole line.

Solution by the Rev. J. WOLSTENHOLME, M.A.

Let $x_1, x_2, \ldots. x_n$ be the distances, in order of magnitude, of the n points of division from one extremity, and 1 the length of the line; then the limiting equation is

$$x_1^2 + (x_2 - x_1)^2 + \ldots. (1 - x_n)^2 < \frac{1}{n},$$

or
$$x_1^2 + \ldots. + x_n^2 - x_1 x_2 - \ldots. - x_{n-1} x_n + \frac{n+1}{2n} < 0.$$

Now I assume as obvious that a point (in space of n dimensions) whose coordinates (rectangular or not) are $x_1, x_2, \ldots. x_n$, is a random point when $x_1, x_2, \ldots. x_n$ are random lengths, and that the chance required is the value of $\iiint \ldots. dx_1 dx_2 \ldots. dx_n$ when the limits are given by the equation

$$x_1^2 + \ldots. + x_n^2 - x_1 x_2 - \ldots. + \frac{n+1}{2n} = 0 \ldots\ldots\ldots\ldots (A),$$

divided by the same integral when the limits are given by $x_1 < x_2 < x_3 \ldots. < x_n < 1$, all positive.

First, to find the quasi-volume of the quadric (A). Its quasi-centre is given by

$$2x_1 - x_2 = 2x_2 - x_1 - x_3 = 2x_3 - x_2 - x_4 = \ldots. = 2x_n - x_{n-1} - 1 = 0,$$

or $\quad x_2 = 2x_1, \quad x_3 = 3x_1 \ldots.,\qquad$ or $\quad x_1 = \dfrac{1}{n+1}, \quad x_2 = \dfrac{2}{n+1} \ldots.$

The equation of the surface referred to its centre is then

$$x_1^2 + \ldots\ldots + x_n^2 - x_1 x_2 - \ldots\ldots - x_{n-1} x_n = \frac{1}{2n(n+1)}.$$

Now, transforming the axes (and to facilitate matters here I assume rectangular axes) so as to destroy the terms involving products, the resulting equation is

$$a_1 X_1^2 + a_2 X_2^2 + \ldots\ldots + a_n X_n^2 = \frac{1}{n(n+1)},$$

where (see my Book of Mathematical Problems, Quest. 921) $a_1, a_2, \ldots\ldots a_n$ are the roots of the equation

$$\begin{vmatrix} z-2, & 1, & 0, & 0, & \ldots\ldots & & \\ 1, & z-2, & 1, & 0, & 0, & \ldots\ldots & \\ 0, & 1, & z-2, & 1, & 0, & 0 \ldots\ldots & \\ \vdots & & & \vdots & & & \vdots \\ 0, & 0, & 0, & \ldots\ldots & 1, & z-2, & 1 \\ 0, & 0, & 0, & \ldots\ldots & 0, & 1, & z-2 \end{vmatrix} = 0,$$

the discriminant of

$$(x_1^2 + \ldots\ldots + x_n^2)\,z - 2\,(x_1^2 + \ldots\ldots + x_n^2 - x_1 x_2 - \ldots\ldots),$$

and the product of these roots is $n+1$. Also the value of $\iiint \ldots dX_1 \ldots dX_n$, where the limits are given by $a_1 X_1^2 + \ldots\ldots + a_n X_n^2 = \dfrac{1}{n(n+1)}$ is (Todhunter's *Integral Calculus*, Art. 274)

$$\frac{1}{(a_1 a_2 \ldots a_n)^{\frac{1}{2}}} \left\{\frac{\pi}{n(n+1)}\right\}^{\frac{1}{2}n} \frac{1}{\Gamma(\frac{1}{2}n+1)} = \frac{1}{(n+1)^{\frac{1}{2}}} \left\{\frac{\pi}{n(n+1)}\right\}^{\frac{1}{2}n} \frac{1}{\Gamma(\frac{1}{2}n+1)};$$

and $\iiint \ldots\ldots dx_1 dx_2 \ldots\ldots dx_n$, over all such positive values that $x_1 < x_2 < x_3 \ldots\ldots < x_n < 1$, is $\dfrac{1}{\lfloor n}$ or $\dfrac{1}{\Gamma(n+1)}$. The chance required is then

$$\frac{1}{(n+1)^{\frac{1}{2}}} \left\{\frac{\pi}{n(n+1)}\right\}^{\frac{1}{2}n} \frac{\Gamma(n+1)}{\Gamma(\frac{1}{2}n+1)}.$$

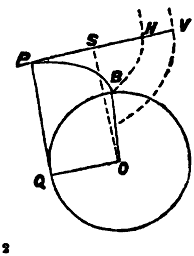
[The Proposer's solution is given on p. 83 of Vol. VI. of the *Reprint*.]

2736. (Proposed by Professor SYLVESTER.)—Find the curve whose radius of curvature and radius vector at each point are equal to one another, and prove that it is the second involute to a circle having an apse midway between the cusp of the first involute (from which it is derived) and the centre of the circle.

Solution by the REV. JAMES WHITE, M.A.

The property of the second involute of a circle stated in this Question, viz., that when its apse is midway between the cusp of the first involute (from

which it is derived) and the centre of the circle, its radius vector and radius of curvature will be equal, may be proved geometrically, as follows :—



Let V be any point on this second involute whose apse is midway between O and B: it is required to prove that $OV = PV$.

The second involute which commences at B, the cusp of the first, will cut PV in H, making $HV = \frac{1}{2}a$, (a being radius of the circle). Let $QP = t$; and draw OS perpendicular to VP; then $OS = t$, and $SP = a$.

Now $\quad OV^2 = OS^2 + VS^2 = t^2 + (VP - a)^2,$

and $\quad VP = HP + \frac{1}{2}a = \text{arc } BP + \frac{1}{2}a = \dfrac{t^2}{2a} + \dfrac{a}{2};$

consequently $\quad OV^2 = t^2 + \left(\dfrac{t^2}{2a} - \dfrac{a}{2}\right)^2 = \left(\dfrac{t^2}{2a} + \dfrac{a}{2}\right)^2;$

therefore $\quad OV = \dfrac{t^2}{2a} + \dfrac{a}{2} = VP;$

that is, radius vector (OV) = radius of curvature (VP).

[Mr. WHITE states that the main part of this proof is due to Mr. CROFTON.]

From this a simple method of finding a point on the second involute corresponding to any one given on the first may be derived.

Bisect the line joining the given point with the centre of the circle, and at the point of bisection erect a perpendicular. Also at the given point erect a perpendicular to the tangent from it to the circle; the intersection of these perpendiculars is a point on the second involute which has its apse midway between the cusp of the first and the centre of the circle (the triangle OVP being isosceles).

Take a point (H) on the perpendicular to the tangent distant from the one found one-half the length the radius of the circle, and it is a point on the second involute which commences at the cusp of the first.

[Professor SYLVESTER sends the following remarks on this question :—

The envelope of a set of half-pitch second involutes to any given circle is a circle concentric with the first. Consequently *any* circle corresponds to a singular solution of the differential equation which defines the curve in question; and thus we obtain another and obviously true solution of the question.]

2764. (Problem by A. SMITH, Q.C.)—It is well known that if A, P, P′ be fixed points on a line (that is, if the distances AP, PP′ are constant), and if P, P′ describe given fixed lines, say the directrices, then the point A describes an ellipse having for its centre the intersection of the two directrices; and the question hence arises: Given three positions of the point A and of the generating line APP′, to find the directrices; or, what is the same thing, Given the lines 1, 2, 3, passing through the points A, B, C respectively, it is required to draw a line cutting the three lines in the points P, Q, R respectively, such that $AP = BQ = CR$.

Solution by ARCHER STANLEY.

Let p, q, r be any three corresponding points on the three given lines; that is to say, let the three segments Ap, Bq, Cr (set off in assigned directions) be equal to one another. Then the connector pq will, by a well-known theorem, envelope a parabola of which Ap is a tangent; in like manner, the connector pr will envelope another parabola having the tangent Ap in common with the first. The remaining two common tangents of these parabolas will manifestly be the two required directrices. The problem is thus reduced to the well-known one (soluble by rule and compass) of finding the common tangents to two parabolas which touch a given line.

2525. (Proposed by G. O. HANLON.)—Three equidistant lines are drawn parallel to an asymptote of a hyperbola, and a triangle inscribed where the lines meet the curve. Then any line parallel to the other asymptote will be divided harmonically by the three sides of the triangle and the curve.

Solution by the Rev. J. WOLSTENHOLME, M.A.; *the* PROPOSER; *and others.*

First to find the locus of a point on any line in a given direction forming a harmonic range with the points where it meets the sides of the triangle of reference.

If $(l+k)x+(m+k)y+(n+k)z=0$ be any such straight line, then for the point (P) in question we have $(m+k)y=(n+k)z$ (if A′B′C′P be harmonic), and the locus is

$$(lx+my+nz)(y-z)+(x+y+z)(nz-my)=0 \quad \ldots\ldots\ldots(1);$$

or, in another form, $\quad 2(m-n)yz-(n-l)zx-(l-m)xy=0 \quad \ldots\ldots\ldots(2).$

Equation (2) shows that this is a conic circumscribing the triangle, and (1) that the asymptotes are parallel to $lx+my+nz=0$ and to $y=z$; the latter of which being the bisector of the side BC, shows that the three straight lines through A, B, C parallel to one asymptote are equidistant. Hence the truth of the theorem.

2727. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—Two lines (a) and (b) in space touch a fixed quadric U, show that a variable line intersecting the two former and touching the latter generates another quadric.

Solution by W. S. BURNSIDE, M.A.

It will be convenient to write the equation of the quadric U under the form $ax^2+by^2+2nxy+2rz\omega=0$, where the planes z and ω plainly touch the surface, and the line (x, y) is their chord of contact. Whence we may consider (x, z) and (y, ω) as the tangent lines (a) and (b). This being so, the condition that a variable line $x=\lambda z$, $\omega=\mu y$, should touch the quadric U is $ab\lambda^2=(n\lambda+\mu r)^2$, giving for the surface traced out by it the equation $abx^2y^2=(nxy+rz\omega)^2$, which resolves itself into the two constituents $(n\pm\sqrt{ab})xy+rz\omega=0$, which represent two quadrics.

2779. (Proposed by Professor SYLVESTER.)—Show that the famous equation between *arc*, *angle of contingence*, and *perpendicular on tangent* to a curve, viz., $p + \frac{d^2p}{d\phi^2} = \frac{ds}{d\phi}$, may be interpreted as affirming the *self-evident proposition* that the radius of curvature is the distance between the tangent at any point of a curve and the tangent at the corresponding point of its second evolute; and account for the connecting sign between p and $\frac{d^2p}{d\phi^2}$ in the above equation being *plus* and not *minus*.

Solution by J. J. WALKER, M.A.

1. Let (r, p, ϕ) refer to the given curve; and let (r_1, p_1, ϕ_1), (r_2, p_2, ϕ_2) refer similarly to its first and second evolutes respectively. Then it is evident that

$$p + p_2 = \rho,$$

where $\rho = \frac{ds}{d\phi}$ is the radius of curvature of the given curve.

Now $\qquad p_1 = \frac{dp}{d\phi}, \quad$ and $\quad p_2 = \frac{dp_1}{d\phi_1} = \frac{dp_1}{d\phi} = \frac{d^2p}{d\phi^2},$

whence

$$p + \frac{d^2p}{d\phi^2} = \rho = \frac{ds}{d\phi}.$$

2. In what precedes, the case has been considered of $\rho > p$. But if $\rho < p$, *i. e.*, $p - p_2 = \rho$, as p increases p_1 decreases, as is evident on drawing the figure; so that $p_2 = -\frac{dp_1}{d\phi_1} = -\frac{d^2p}{d\phi^2}$. Hence the sign between p and $\frac{d^2p}{d\phi^2}$ is in both cases *plus*.

2719. (Proposed by Professor WHITWORTH.)—1. The six points which, in conjunction with any common transversal, divide harmonically the six sides of any tetrastigm, lie all on one conic, which also passes through the three points of intersection of opposite sides of the tetrastigm.

2. Of the six points, if those pairs which lie on opposite sides of the tetrastigm be joined by straight lines, the three straight lines thus drawn are concurrent, and their point of intersection is the pole of the transversal with respect to the conic.

I. *Solution by* W. S. MCCAY, B.A.

It is not hard to prove this directly by Pascal's theorem; but it is easier to consider the line at infinity, then we know (1) that the locus of centres of conics through the four points is a conic through the six middle points of the lines of the system, and that three other points on the locus are the three intersections of opposite lines of the system, and (2) that the centre of this conic is the intersection of the three lines joining the middle points of opposite lines; for these three lines meet and bisect each other in the centre of gravity of equal weights at the four vertices of the tetrastigm.

II. *Solution by* F. D. THOMSON, M.A.

1. This follows at once from the well known theorem, that if a system of conics be described about a given quadrilateral, the locus of the pole of a fixed line with respect to these conics is another conic.

This theorem may be proved analytically, as in Salmon; or geometrically, as follows:

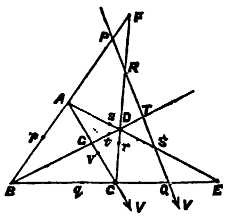
The given transversal is cut in points in involution by any conic of the system, and by the opposite sides of the quadrilateral; and the *double points* of the involution are the points where two conics of the system *touch* the transversal.

Now, for each of the points where the transversal cuts the required locus, the pole lies on its polar, and therefore must be the point of contact of the transversal with a conic of the system.

Hence the *only points* in which the transversal meets the required locus are the two double points of the involution determined by the intersections of the transversal and the opposite sides and diagonals of the quadrangle.

Hence the required locus is a *conic section*, and it is easily seen to pass through the points mentioned in the Question.

2. The former part of this proposition may be derived by projection from well known properties of the *middle* points of the sides of a quadrilateral; or *directly*, as follows:



Let the transversal meet the sides in P, Q, R, S, T, V; and let the points of harmonic section be p, q, r, s, t, v. Then

$$\{\mathrm{B}p\mathrm{AP}\} = \{\mathrm{D}s\mathrm{AS}\};$$

therefore $$\mathrm{T}\{\mathrm{B}p\mathrm{AP}\} = \mathrm{T}\{\mathrm{D}s\mathrm{AS}\}$$
$$= \mathrm{T}\{\mathrm{B}s\mathrm{AP}\};$$

therefore Tps is a straight line. Similarly, Tqr, Vrs, Vqp, Rsv, Qpv, &c. are straight lines. Hence, by the harmonic properties of a quadrilateral, pr and sq intersect in the pole of TV with respect to any conic through $pqrs$, and therefore the intersection of pr and sq is the pole of the given transversal with regard to the conic of the Question. Similarly, the intersection of sq and vt is the pole of PR, *i. e.* of the transversal, with respect to the same conic. Hence sq, pr, vt meet in the pole of the transversal.

2766. (Proposed by S. WATSON.)—Three points move from a fixed point in three different but given directions, with given uniform but different velocities. Find the locus of the centre of the circle passing through them.

I. *Quaternion Solution by* W. H. LAVERTY, B.A.

Let α, β, γ be the unit-vectors in the directions of motion, and at the end of any time let $kx_1\alpha$, $kx_2\beta$, $kx_3\gamma$ be the vectors to the moving points; so that, ρ being the vector to the centre of the circle, we have

$$(kx_1\alpha-\rho)^2 = (kx_2\beta-\rho)^2 = (kx_3\gamma-\rho)^2,$$

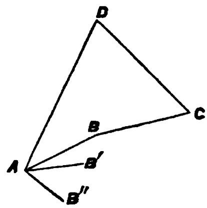
therefore $kx_1^2 + 2x_1 S\alpha\rho = kx_2^2 + 2x_2 S\beta\rho = kx_3^2 + 2x_3 S\gamma\rho$;
hence, eliminating k, and making obvious substitutions, we get

$$S(a\alpha + b\beta + c\gamma)\rho = 0,$$

where $a+b+c = 0$; and therefore the locus is a straight line perpendicular to the vector

$$\rho' = x(a\alpha + b\beta + c\gamma).$$

If AB, BC, DC be the directions respectively of α, β, γ; and if, arithmetically, AB + BC = CD; then AD is the direction of ρ'.

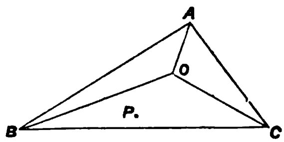


II. *Quaternion Solution by* F. D. Thomson, M.A.; *and* G. A. Ogilvie.

It is easy to see that the locus must be a straight line through the origin, since the triangle formed by the moving points always remains parallel and similar to its initial position; but it may be interesting to give an investigation of the locus by the method of *Quaternions*.

Let O be the point of departure; A, B, C the points at the time t; u, v, w vectors representing the constant velocities of the points. Then, writing α, β, γ for OA, OB, OC, we have

$$\frac{\alpha}{u} = \frac{\beta}{v} = \frac{\gamma}{w} = t.$$



Now, let P be the centre of the circle circumscribing ABC, and let OP $= \rho$.

Then we have to express the facts, first, that P is in the plane ABC, and second, that it is equidistant from ABC.

The first condition is equivalent to the assertion that PA, AB, BC are in one plane, or that $S.PA.AB.BC = 0$, or $S.(\alpha-\rho)(\beta-\alpha)(\gamma-\beta) = 0$, or $S.\rho(V\alpha\beta + V\beta\gamma + V\gamma\alpha) = S\alpha\beta\gamma$, or $S.\rho(Vuv + Vvw + Vwu) = tSuvw$ ... (1).

The second condition gives

$$(\alpha-\rho)^2 = (\beta-\rho)^2 = (\gamma-\rho)^2, \text{ or } \alpha^2 - 2S\rho\alpha = \beta^2 - 2S\rho\beta = \gamma^2 - 2S\rho\gamma.$$

or $\qquad 2S\rho(\beta-\alpha) = \beta^2 - \alpha^2$, and $2S\rho(\gamma-\beta) = \gamma^2 - \beta^2$;

or $\qquad 2S\rho(v-u) = t(v^2-u^2)$, and $2S\rho(w-v) = t(w^2-v^2)$ ...... (2, 3).

Eliminating t,

$$\frac{S.\rho(Vuv + Vvw + Vwu)}{Suvw} = \frac{2S\rho(v-u)}{v^2-u^2} = \frac{2S\rho(w-v)}{w^2-v^2},$$

a system equivalent to two equations, each of the form $S\rho\delta = 0$, and therefore representing two planes through the origin, their intersection being the locus required.

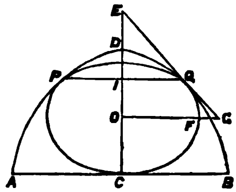
Similarly, it may be shown that the locus of the intersection of perpendiculars of the triangle ABC has for its equations

$$\frac{S.\rho(Vuv + Vvw + Vwu)}{Suvw} = \frac{S\rho(w-v)}{Su(w-v)} = \frac{S\rho(u-w)}{Sv(u-w)}.$$

[The three directions of motion are, in the *first* solution, supposed to be in one plane, but in the *second* solution, anywhere in space.]

2745. (Proposed by the EDITOR.)—Show that the average area of all the ellipses that can be inscribed symmetrically in a portion of a parabola cut off by a double ordinate perpendicular to the axis, in parts of the area of the parabola, is $\frac{23}{160}\pi$ if (1) the centres be equably distributed over all possible positions on the axis, and $\frac{11}{64}\pi$ if (2) the intersections of the chords of contact with the axis be equably distributed on the axis.

I. *Solution by* STEPHEN WATSON.



Let AB be the double ordinate, CD the axis, PQ the chord of contact cutting CD in I, O the centre of the inscribed ellipse, and EG a common tangent at Q, meeting CD produced in E, and the semi-diameter OF, perpendicular to CD produced, in G.

Put $CD = a$, $CB = b$, $CO = \alpha$, $OF = \beta$, $OI = ax$; then we have
$\alpha^2 = OE.OI = (2ID + OI)\,OI = 2(a-\alpha)\,ax - a^2x^2$,

therefore $\alpha = a\left\{(2x)^{\frac{1}{2}} - x\right\}$, and $EI\ (=2DI) : IQ = EO : OG$,

therefore $$\beta^2 = OG.IQ = \frac{EO.IQ^2}{2DI} = \frac{b^2}{2a}\,EO = \frac{b^2\alpha^2}{2a^2x};$$

therefore $$\beta = b\left\{1 - \left(\frac{x}{2}\right)^{\frac{1}{2}}\right\};$$

hence the area of the ellipse in parts of that of the parabola is

$$\frac{\pi\alpha\beta}{\frac{4}{3}ab} = \frac{3\pi}{4}\left\{(2x)^{\frac{1}{2}} - x\right\}\left\{1 - \left(\frac{x}{2}\right)^{\frac{1}{2}}\right\} \;\ldots\ldots\ldots\ldots\ldots \text{(A)}.$$

(1). In this case the number of possible positions of O is $\frac{1}{2}a$, and the limits of x are 0 and $\frac{1}{2}$; hence the required average is

$$\frac{2}{a}\int_0^{\frac{1}{2}a} (\text{A})\,d\alpha = \frac{3\pi}{2}\int_0^{\frac{1}{2}} \left\{(2x)^{\frac{1}{2}} - x\right\}\left\{1 - \left(\frac{x}{2}\right)^{\frac{1}{2}}\right\}\left\{(2x)^{-\frac{1}{2}} - 1\right\}dx = \frac{23}{160}\pi.$$

(2). In this case the number of positions of I is a; and the limits of x, 0 and $\frac{1}{2}$; hence the required average is

$$\frac{1}{a}\int_0^{a} (\text{A})\,d(\alpha + ax) = \frac{3\pi}{4}\int_0 \left\{(2x)^{\frac{1}{2}} - x\right\}\left\{1 - \left(\frac{x}{2}\right)^{\frac{1}{2}}\right\}(2x)^{-\frac{1}{2}}dx = \frac{11}{64}\pi.$$

II. *Solution by the* PROPOSER.

Let the equation to the parabola be

$$y^2 = 4mx \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)};$$

then, (α, β) being the semi-axes of an inscribed ellipse, its equation will be

$$\frac{(x - a + \alpha)^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1 \;\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}.$$

Eliminating y from (1) and (2), and applying the condition for equal roots to the resultant, we find that the ellipse (2) will have double symmetric contact with the given parabola (1) if

$$\beta^2 + 2m\alpha = b\beta, \quad \text{or} \quad \alpha = \frac{b\beta - \beta^2}{2m} \quad \ldots\ldots\ldots\ldots\ldots (3).$$

Otherwise: The equation of a conic which has double contact with the parabola (1), along the chord PQ or $x - z = 0$ where $z = \text{DI}$, is

$$y^2 - 4mx = k(x-z)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4);$$

and determining k so that (4) may touch the double ordinate AB, we find

$$k = -\frac{4ma}{(a-z)^2} = -\frac{b^2}{(a-z)^2};$$

hence the equation of an ellipse inscribed in the parabola becomes

$$\frac{a}{m(a+z)^2}\,y^2 + \frac{4a^2}{(a^2-z^2)^2}\left(x - \frac{a^2+z^2}{2a}\right)^2 = 1 \quad \ldots\ldots\ldots (5);$$

and its semi-axes are $\quad \alpha = \dfrac{a^2-z^2}{2a}, \quad \beta = \dfrac{2m(a+z)}{b} \quad \ldots\ldots\ldots\ldots\ldots (6).$

From (3) and (6) we see that the area of the ellipse, in parts of the parabola, is

$$\frac{3\pi}{2b^3}(b\beta^2 - \beta^3), \quad \text{or} \quad \frac{3\pi}{16a^3}(a+z)(a^2-z^2) \ldots\ldots\ldots (7, 8).$$

In (1), α varies uniformly from 0 to $\frac{1}{2}a$; hence, by (7), the average area of the inscribed ellipses is

$$\frac{3\pi}{ab^3}\int_0^{\frac{1}{2}a}(b\beta - \beta^3)\,d\beta = \frac{6\pi}{b^5}\int_b^b (b\beta - \beta^3)(b - 2\beta)\,d\beta$$

$$= \frac{6\pi}{b^5}\int_{\frac{1}{2}b}^{b}(-2\beta^4 + 3b\beta^3 - b^2\beta^2)\,d\beta = \frac{23}{160}\pi.$$

In (2), z varies uniformly from 0 to a; hence, by (8), the average area of the inscribed ellipses is

$$\frac{3\pi}{16a^4}\int_0^a (a^3 + a^2z - az^2 - z^3)\,dz = \frac{11}{64}\pi.$$

From (7) or (8) we find that the inscribed ellipse is a maximum when $z = \frac{1}{3}a$, $\alpha = \frac{4}{9}a$, $\beta = \frac{2}{3}b$; its area being $\frac{2}{9}\pi$.

2640. (Proposed by the Rev. J. Wolstenholme, M.A.)—1. If a conic of given eccentricity e circumscribe a triangle ABC, the locus of its centre is

$$\frac{(yz\sin^2 A + zx\sin^2 B + xy\sin^2 C)^2}{xyz\,(x+y+z)\sin^2 A\sin^2 B\sin^2 C} = \frac{(e^2-2)^2}{1-e^2};$$

and (2) if it be inscribed in the triangle, the locus of the centre is

$$\frac{(x^2 \sin^2 A + \dots + 2yz \sin B \sin C \cos A + \dots)^2}{xyz\,(x+y+z) \sin^2 A \sin^2 B \sin^2 C} = \frac{4\,(e^2-2)^2}{1-e^2};$$

the triangle of reference having its angular points at the middle points of the sides of the triangle ABC.

3. Hence show that the locus of points with which as centre two similar conics can be drawn, one inscribed and the other circumscribed to the triangle, is two circles, one the circumscribed circle of the triangle, and the other that which has the centre of gravity and the centre of perpendiculars of the triangle as the ends of a diameter.

I. *Solution by* W. S. M'Cay, B.A., *and the* Proposer.

1. If Σ and Ω be the tangential equations of a conic and the circular points at infinity; then the values of k^2, for which $\Sigma + k^2\Omega = 0$ represents points, are proportional to the squares of the axes of the conic, and are given by the discriminating equation

$$\Delta^2 + \Delta\Theta' k^2 + \Theta k^4 = 0,$$

for this is evidently so in the case $a^2\lambda^2 + b^2\mu^2 - \nu^2 + k^2(\lambda^2+\mu^2) = 0$.

Hence, easily, if ϕ be the angle between the asymptotes

$$\frac{(e^2-2)^2}{e^2-1} = 4\cot^2\phi = \frac{\Theta'^2}{-\Theta},$$

Θ being positive for an ellipse and negative for an hyperbola.

Now for the circumscribing conic $2\,(fyz + gxz + hxy) = 0$,

$$-\Theta = f^2 \sin^2 A + \dots - 2gh \sin B \sin C \dots,$$

$$-\Theta' = 2\,(f \cos A + g \cos B + h \cos C).$$

But $$f : g : h = \frac{X}{\sin A}(-X+Y+Z) : (\ \ ) : (\ \ ),$$

in terms of areal coordinates of centre,

or $$f : g : h = \frac{y+z}{\sin A}\,x \ : \ \frac{x+z}{\sin B}\,y \ : \ \frac{x+y}{\sin C}\,z,$$

the coordinates being referred to the triangle whose vertices are the middle points of the sides of the original triangle. Hence we obtain at once the first equation given in the question.

2. Again, for the inscribed conic $(lx)^{\frac12} + (my)^{\frac12} + (nz)^{\frac12} = 0$,

$$\Theta = 2lmn\,(l \sin B \sin C + m \sin A \sin C + n \sin A \sin B),$$

$$\Theta' = l^2 + m^2 + n^2 + 2mn \cos A + \dots,$$

and $l : m : n = \sin A\,(-X+Y+Z) : (\ \ ) : (\ \ ) = x \sin A : y \sin B : z \sin C.$

Hence we obtain the second equation given in the question.

3. Eliminating e from the equations in (1) and (2), we see that if with the same point (x, y, z) as centre there can be drawn two similar conics, one inscribed in the triangle ABC and the other circumscribed about it, the point (x, y, z) must lie on one of the two circles

$$x^2 \sin^2 A + \dots + 2yz \sin B \sin C \cos A + \dots = \pm 2\,(yz \sin^2 A + \dots).$$

If the lower sign be taken, this is the circumscribed circle of the triangle ABC; and if the upper, it is the circle in which the centre of perpendiculars and the centre of gravity are ends of a diameter. This latter circle is therefore one of the system coaxal with the circumscribed and nine-point circles.

II. *Solutions by the* PROPOSER.

1. The equation of a conic circumscribing the triangle ABC, referred to AB, AC as axes of Cartesian coordinates, is of the form

$$\lambda X(X-c)+\mu XY+\nu Y(Y-b)=0;$$

and if e be the eccentricity,

$$\frac{(e^2-2)^2}{1-e^2}=\frac{4(\lambda+\nu-\mu\cos A)^2}{\sin^2 A(4\nu\lambda-\mu^2)};$$

also for its centre $\quad \lambda(2X-c)+\mu Y=\nu(2Y-b)+\mu X=0;$

but if we transfer to areal coordinates measured on the triangle abc (the middle points of the sides of ABC),

$$2\frac{X}{c}=z+x,\quad 2\frac{Y}{b}=y+x;$$

and therefore for the centre

$$\frac{\lambda c^2}{(x+y)z}=\frac{\nu b^2}{y(x+z)}=\frac{\mu bc}{2yz},$$

and the locus of the centre, when the eccentricity e is given, is

$$\frac{(e^2-2)^2}{1-e^2}=\frac{\left\{z(x+y)\sin^2 B+y(x+z)\sin^2 C-2yz\sin B\sin C\cos A\right\}^2}{\sin^2 A\sin^2 B\sin^2 C\left\{yz(x+y)(x+z)-y^2z^2\right\}},$$

which easily reduces to the form given in the question.

2. The equation of an inscribed conic is

$$\frac{X}{h}+\frac{Y}{k}-1=2\lambda\left(\frac{XY}{hk}\right)^{\frac{1}{2}},$$

with the condition $\quad \left(\frac{1}{h}-\frac{1}{c}\right)\left(\frac{1}{k}-\frac{1}{b}\right)=\frac{\lambda^2}{hk};$

and for the centre $\quad \frac{X}{h}=\frac{Y}{k}=\frac{1}{2(1-\lambda^2)};$

and combining these, we find

$$\frac{h}{X}=\frac{k}{Y}=2(1-\lambda^2)=\frac{bc}{XY}\left(\frac{X}{c}+\frac{Y}{b}-\frac{1}{2}\right).$$

Transforming as before to the triangle abc, we find

$$\lambda^2=\frac{yz}{(x+y)(x+z)},\quad \frac{h}{c}(x+y)=\frac{k}{b}(x+z)=x(x+y+z).$$

But, for the eccentricity e, we have the equation

$$\frac{(e^2-2)^2}{1-e^2}=\frac{\left\{\frac{1}{h^2}+\frac{1}{k^2}-\frac{2}{hk}(1-2\lambda^2)\cos A\right\}^2}{\sin^2 A\left\{\frac{1}{h^2k^2}-\frac{(1-2\lambda^2)^2}{h^2k^2}\right\}}$$

$$\equiv\frac{\left\{(x+y)^2\sin^2 B+(x+z)^2\sin^2 C-2(x+y)(x+z)\sin B.\sin C\cos A+4\sin B\sin C\cos A\,yz\right\}^2}{4\sin^2 A\sin^2 B\sin^2 C.yz.x(x+y+z)}$$

$$=\frac{(x^2\sin^2 A+\ldots\ldots+2yz\sin B\sin C\cos A+\ldots\ldots)^2}{4xyz(x+y+z)\sin^2 A\sin^2 B\sin^2 C}.$$

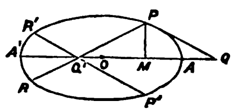
Another solution may be obtained in the following manner:—

1. If the triangle of reference be the triangle having its angles at the middle points of the sides of the triangle ABC, the equation of a conic circumscribing ABC is
$$l(x+y)(x+z)+m(y+z)(y+x)+n(z+x)(z+y)=0,$$
and the equation of the straight lines through A parallel to its asymptotes is
$$lyz-mz(y+z)-ny(y+x)=0\,;$$
whence, if ϕ be the angle between the asymptotes,
$$\tan^2\phi=\frac{\sin^2 A\sin^2 B\sin^2 C\left\{(m+n-l)^2-4mn\right\}}{\left\{n\sin^2 B-(m+n-l)\sin B\sin C\cos A+m\sin^2 C\right\}^2}.$$
Now, if e be the eccentricity and (X, Y, Z) the centre,
$$\frac{(e^2-2)^2}{1-e^2}=-4\cot^2\phi,\quad\text{and}\quad\frac{l}{X(Y+Z)}=\frac{m}{Y(Z+X)}=\frac{n}{Z(X+Y)}\,;$$
whence, if e be given, the locus of the centre is
$$\frac{1-e^2}{(e^2-2)^2}=\frac{\sin^2 A\sin^2 B\sin^2 C\left\{YZ(Z+X)(X+Y)-Y^2Z^2\right\}}{\left\{Z(X+Y)\sin^2 B-2YZ\sin B\sin C\cos A+Y(Z+X)\sin^2 C\right\}^2}$$
$$\equiv\frac{XYZ(X+Y+Z)\sin^2 A\sin^2 B\sin^2 C}{\left\{YZ\sin^2 A+ZX\sin^2 B+XY\sin^2 C\right\}^2}.$$

2. With a similar notation, the equation of an inscribed conic being
$$\left\{l(y+z)\right\}^{\frac12}+\left\{m(z+x)\right\}^{\frac12}+\left\{n(x+y\right\}^{\frac12}=0,$$
$$\tan^2\phi=-\frac{16lmn(l+m+n)\sin^2 A\sin^2 B\sin^2 C}{\left\{l^2\sin^2 A+m^2\sin^2 B+n^2\sin^2 C+2mn\sin B\sin C\cos A+\ldots+\ldots\right\}^2},$$
and $\dfrac{X}{l}=\dfrac{Y}{m}=\dfrac{Z}{n}$, whence the locus of the centre of an inscribed conic of given eccentricity e is
$$\frac{(e^2-2)^2}{1-e^2}=\frac{(X^2\sin^2 A+\ldots+2YZ\sin B\sin C\cos A+\ldots)^2}{4XYZ(X+Y+Z)\sin^2 A\sin^2 B\sin^2 C}.$$

2781. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—Prove that the maximum value of the common chord of an ellipse and its circle of curvature, a and b being the semiaxes of the ellipse, is
$$\frac{4}{3\sqrt{3}\,(a^2-b^2)}\left\{2\,(a^4-a^2b^2+b^4)^{\frac32}-(a^2+b^2)(2a^2-b^2)(2b^2-a^2)\right\}^{\frac12}.$$

I. *Solution by* MATTHEW COLLINS, B.A.



Let $AA'=2a$ be the major axis, O the centre, and $PQ'=t=$ the tangent PQ; then $PQ'R$ is well known to be a chord of the circle osculating the ellipse at P. Let $OM=x$, the perpendicular $MP=y$; then we have

$$OQ = \frac{a^2}{x}, \quad MQ = MQ' = \frac{a^2}{x} - x, \quad t^2 = \left(\frac{a^2}{x} - x\right)^2 + y^2 = \left(\frac{a^2}{x} - x\right)^2 + b^2 - \frac{b^2x^2}{a^2}.$$

Again, drawing P′Q′R′ parallel to PQ, and putting $Q'R = z$, we have

$$\frac{QA \,.\, QA'}{QP^2} = \frac{Q'A \,.\, Q'A'}{Q'P' \,.\, Q'R'} = \frac{Q'A \,.\, Q'A'}{Q'P \,.\, Q'R}, \quad i.e. \quad \frac{\frac{x^4}{a^2} - a^2}{t^2} = \frac{a^2 - \left(\frac{a^2}{x} - 2x\right)^2}{tz},$$

$$\text{therefore} \qquad \frac{z}{t} = \frac{a^2 - \left(\frac{a^2}{x} - 2x\right)^2}{\frac{a^4}{x^2} - a^2} = \frac{(a^2 - x^2)(4x^2 - a^2)}{a^2(a^2 - x^2)} = \frac{4x^2 - a^2}{a^2};$$

$$\text{therefore} \qquad PR = t + z = t\left(1 + \frac{z}{t}\right) = t\left(\frac{4x^2}{a^2}\right).$$

Now, when this is a maximum,

$$x^4t^2 = x^4\left\{\left(\frac{a^2}{x} - x\right)^2 + b^2 - \frac{b^2x^2}{a^2}\right\} = \frac{1}{a^2}\left\{(a^2 - b^2)x^6 - a^2(2a^2 - b^2)x^4 + a^6x^2\right\}$$

must also be a maximum; its differential coefficient equated to zero gives

$$3(a^2 - b^2)x^4 - 2a^2x^2(2a^2 - b^2) + a^6 = 0, \quad \text{whence} \quad \frac{a^4}{x^2} = 2a^2 - b^2 \pm R = OQ^2,$$

where $R = (a^4 - a^2b^2 + b^4)^{\frac{1}{2}}$ is plainly $> a^2 - \frac{1}{2}b^2$; and therefore, as $OQ^2 > a^2$, the *proper* value of OQ^2 is $2a^2 - b^2 + R$; hence

$$t^2 = \frac{a^4}{x^2} - 2a^2 + b^2 + \frac{x^2}{a^2}(a^2 - b^2) = R + \frac{a^2(a^2 - b^2)}{2a^2 - b^2 + R}$$

$$= R + \frac{2a^2 - b^2 - R}{3} = \frac{2R + 2a^2 - b^2}{3};$$

$$\text{therefore } PR^2 = \frac{16}{a^4}t^2x^4 = \frac{16a^4t^2}{\left(\frac{a^4}{x^2}\right)^2} = \frac{16a^4}{3} \cdot \frac{2R + 2a^2 - b^2}{(R + 2a^2 - b^2)^2}$$

$$= \frac{16a^4}{3} \cdot \frac{(2R + 2a^2 - b^2)(2a^2 + b^2 - R)^2}{\left\{3a^2(a^2 - b^2)\right\}^2}$$

$$= \frac{16}{27(a^2 - b^2)^2}\left\{2R^3 - 3R^2(2a^2 - b^2) + (2a^2 - b^2)^3\right\}$$

$$= \frac{16}{27(a^2 - b^2)^2}\left\{2R^3 - (2a^2 - b^2)\left[3R^2 - (2a^2 - b^2)^2\right]\right.$$

$$= \frac{16}{27(a^2 - b^2)^2}\left\{2(a^4 - a^2b^2 + b^4)^{\frac{3}{2}} - (a^2 + b^2)(2a^2 - b^2)(2b^2 - a^2)\right\},$$

the required expression for the square of the maximum common chord.

II. *Solution by the* PROPOSER.

The circle of curvature of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, at the point whose eccentric angle is θ, meets the ellipse again in the point $(a\cos 3\theta, \ -b\sin 3\theta)$, and the square on the common chord is therefore

$$u^2 = 4\sin^2 2\theta\,(a^2\sin^2\theta + b^2\cos^2\theta) = 2(1-z^2)(a^2+b^2-c^2z), \text{ if } z \equiv \cos 2\theta.$$

For a maximum or minimum value,

either $$\frac{dz}{d\theta} = 0, \quad\text{or}\quad 3c^2z^2 - 2(a^2+b^2)z - c^2 = 0,$$

of which the former gives $u=0$; and since u^2 is essentially positive, this is the minimum value. The maximum value is then given by that root of the latter equation which is numerically less than 1, or by

$$z = \frac{a^2+b^2-2(a^4-a^2b^2+b^4)^{\frac{1}{2}}}{3(a^2-b^2)} \equiv \frac{a^2+b^2-2\text{R}}{3c^2}.$$

For this value we have

$$u^2 = \frac{2z}{3}\left\{2(a^2+b^2)z+c^2\right\} - \frac{2(a^2+b^2)}{3c^2}\left\{2(a^2+b^2)z+c^2\right\} - 2c^2z + 2(a^2+b^2)$$

$$= \frac{4(a^2+b^2)}{9c^2}\left\{2(a^2+b^2)z+c^2\right\} - \frac{4(a^2+b^2)}{3c^2}z - \frac{4c^2z}{3} + \frac{4(a^2+b^2)}{3}.$$

$$= -\frac{4}{9c^2}z\left\{(a^2+b^2)^2+3(a^2-b^2)^2\right\} + \frac{16}{9}(a^2+b^2)$$

$$= -\frac{16}{27c^4}(a^4-a^2b^2+b^4)(a^2+b^2-2\text{R}) + \frac{16}{9}(a^2+b^2)$$

$$= \frac{16}{27c^4}\left\{2\text{R}^3-(a^2+b^2)\left[(a^4-a^2b^2+b^4)-3(a^2-b^2)^2\right]\right\}$$

$$= \frac{16}{27(a^2-b^2)^2}\left\{2(a^4-a^2b^2+b^4)^{\frac{3}{2}}-(a^2+b^2)(2a^2-b^2)(2b^2-a^2)\right\}.$$

The longest common chord has therefore the value given in the question.

2345. (Proposed by H. R. GREER, B.A.)—Determine in trilinear coordinates the foot of the perpendicular from the point (x', y', z') on the line $lx+my+nz=0$; also the reflexion of the same point with regard to the same line.

Solution by JAMES DALE.

If the line $\quad \text{L}x+\text{M}y+\text{N}z=0 \quad\ldots\ldots\ldots\ldots\ldots(1)$

be perpendicular to $\quad lx+my+nz=0 \quad\ldots\ldots\ldots\ldots\ldots(2),$

and pass through (x', y', z'), we have the conditions

$$\text{L}x'+\text{M}y'+\text{N}z'=0,\quad \text{L}l'+\text{M}m'+\text{N}n'=0$$

(where $l' = l - n\cos\text{B} - m\cos\text{C}$, with corresponding values for m', n');

therefore $$\frac{\text{L}}{\begin{vmatrix} m' & n' \\ y' & z'\end{vmatrix}} = \frac{\text{M}}{\begin{vmatrix} n' & l' \\ z' & x'\end{vmatrix}} = \frac{\text{N}}{\begin{vmatrix} l' & m' \\ x' & y'\end{vmatrix}}.$$

The intersection of (1) and (2), that is, the foot of the perpendicular from the given point on the given line, is

$$\frac{\xi}{\begin{vmatrix} m & n \\ M & N \end{vmatrix}} = \frac{\eta}{\begin{vmatrix} n & l \\ N & L \end{vmatrix}} = \frac{\zeta}{\begin{vmatrix} l & m \\ L & M \end{vmatrix}};$$

and substituting for L, M, N, we get

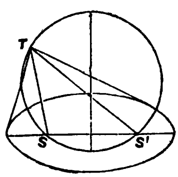
$$\frac{\xi}{\begin{vmatrix} m & n \\ \begin{vmatrix} n' & l' \\ z' & x' \end{vmatrix} & \begin{vmatrix} l' & m' \\ x' & y' \end{vmatrix} \end{vmatrix}} = \frac{\eta}{\begin{vmatrix} n & l \\ \begin{vmatrix} l' & m' \\ x' & y' \end{vmatrix} & \begin{vmatrix} m' & n' \\ y' & z' \end{vmatrix} \end{vmatrix}} = \frac{\zeta}{\begin{vmatrix} l & m \\ \begin{vmatrix} m' & n' \\ y' & z' \end{vmatrix} & \begin{vmatrix} n' & l' \\ z' & x' \end{vmatrix} \end{vmatrix}}$$

$$= \frac{2\Delta}{\begin{vmatrix} a & b & c \\ l & m & n \\ \begin{vmatrix} m' & n' \\ y' & z' \end{vmatrix} & \begin{vmatrix} n' & l' \\ z' & x' \end{vmatrix} & \begin{vmatrix} l' & m' \\ x' & y' \end{vmatrix} \end{vmatrix}}.$$

The coordinates of the point of reflexion (x_1, y_1, z_1) may now be found from the equations $\quad 2\xi = x' + x_1, \quad 2\eta = y' + y_1, \quad 2\zeta = z' + z_1.$

2301. (Proposed by W. K. CLIFFORD.)—A circle is drawn so that its radical axis with respect to the focus S of a parabola is a tangent to the parabola; if a tangent to the circle cut the parabola in A, B, and if SC, bisecting the angle ASB, cut AB in C, the locus of C is a straight line.

Solution by the REV. J. WOLSTENHOLME, M.A.



Through the foci S, S′ of a conic draw a circle, and from any point T on this circle draw two tangents; the straight lines bisecting the angles between these tangents will bisect the angles between TS and TS′, and will therefore pass through the fixed points where the circle meets the minor axis. Reciprocate the system on S; then to the circle corresponds a parabola having S for its focus, and to the ellipse a circle having for radical axis to it and the point-circle S a tangent to the parabola (the reciprocal of S′); to the point T corresponds a tangent to the parabola; to the tangents the two points A, B; and to the straight lines bisecting the angle, points on AB where the straight lines bisecting ASB meet it. Whence the theorem.

2456. (Proposed by Dr. SHAW.)—Show that if an ellipse pass through the centre of a hyperbola, and have its foci on the hyperbola's asymptotes; (*a*) the hyperbola passes through the centre of the ellipse, (*b*) the axes of each curve are respectively tangent and normal to each other, and (*c*) the two axes which are also normals are equal to each other.

Solution by W. H. LAVERTY, B.A.

(*a*) Let the equation to the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; therefore the equations to the asymptotes of the hyperbola are

$$xy'-y(x'-c)-cy'=0, \text{ and } xy'-y(x'+c)+cy'=0,$$

where the centre (x', y') of the hyperbola is on the ellipse. Hence the equation to the hyperbola is

$$x^2y'^2+y^2(x'^2-c^2)-2xy\,x'y'+2c^2yy'+k^4=0,$$

where k^4 is a constant. The curve then will not pass through the centre of the ellipse unless $k=0$, when the equation becomes

$$x^2y'^2+y^2(x'^2-c^2)-2xy\,x'y'+2c^2yy'=0\,..............\,(1).$$

(*b*.) If, in (1), we put $y=0$, we have $x^2=0$; therefore the major axis of the ellipse is a tangent to the hyperbola at the origin, and the minor axis is a normal; also the axes of (1), being the bisectors of the focal distances, are tangent and normal to the ellipse.

(*c*.) Again, transform the equation (1) so that (x', y') may be the origin, and the tangent and normal to the ellipse the coordinate axes; the transformed equation will be
$$\frac{x^2}{b^2}-\frac{y^2}{c^2-e^2x'^2}=1.$$
Therefore the axis of the hyperbola, which is a normal to the ellipse, is equal to b, and is equal to the minor axis of the ellipse, which, by (*b*), is the normal to the hyperbola.

2653. (Proposed by J. GRIFFITHS, M.A.)—1. If the centre of one of the four circles which touch the sides of a triangle self-conjugate with respect to a parabola lies on the directrix; show that its circumference passes through the focus.

2. If the point of intersection of the three perpendiculars of a triangle inscribed in a parabola lies on the directrix; show that the self-conjugate circle of this triangle passes through the focus of the curve.

Solution by the REV. J. WOLSTENHOLME, M.A.

1. If we take the triangle as that of reference, the equation of the parabola is $lx^2+my^2+nz^2=0$, with the condition $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}=0$.

The equation of the directrix is $x\left(\frac{b^2}{n}+\frac{c^2}{m}\right)+\ldots.+\ldots.=0$;

which will pass through the centre of the inscribed circle, if

$$a\left(\frac{b^2}{n}+\frac{c^2}{m}\right)+\ldots.+\ldots.=0, \text{ or } \frac{b+c}{al}+\ldots.+\ldots.=0.$$

Hence $l\left(\frac{c+a}{b}-\frac{a+b}{c}\right)=m\left(\frac{a+b}{c}-\frac{b+c}{a}\right)=n\left(\frac{c+a}{b}-\frac{a+b}{c}\right)$,

or $$la(b-c)=mb(c-a)=nc(a-b).$$

But the focus is given by the equations

$$\frac{x}{mb^2+nc^2} = \ldots., \quad \text{or} \quad \frac{x}{\dfrac{b}{c-a}+\dfrac{c}{a-b}} = \ldots. = \ldots.$$

or
$$\frac{x}{(b-c)^2(b+c-a)} = \frac{y}{(c-a)^2(c+a-b)} = \frac{z}{(a-b)^2(a+b-c)};$$

and this point manifestly lies on the inscribed circle $(x\cot\frac{1}{2}A)^{\frac{1}{2}}+\ldots+\ldots=0$. This point is of course the point of contact of the inscribed, or escribed, circle with the nine-point circle. It may be noticed that the tangent to this circle at the focus of the parabola is the polar with respect to the parabola of the centre of the circle, its equation being $\dfrac{x}{b-c}+\dfrac{y}{c-a}+\dfrac{z}{a-b}=0$.
There is no difference in the work when one of the other circles is taken, beyond changing the signs of one of the three, a, b, c.

2. This is only the reciprocal of the above proposition with respect to the focus of the parabola.

2718. (Proposed by Professor CAYLEY.)—Find *in plano* the locus of a point P, such that from it two given points A, C, and two given points B, D, subtend equal angles.

2757. (Proposed by Professor CAYLEY.)—If

$$x_0^2+y_0^2=1, \quad x_1^2+y_1^2=1, \quad \text{and} \quad \begin{vmatrix} x, & y, & 1 \\ x_0, & y_0, & 1 \\ x_1, & y_1, & 1 \end{vmatrix} = L;$$

show that each of the equations

$$\frac{a^2(x-x_0)^2+b^2(y-y_0)^2}{(xx_0+yy_0-1)^2} = \frac{a^2(x-x_1)^2+b^2(y-y_1)^2}{(xx_1+yy_1-1)^2} \ldots\ldots\ldots (1),$$

$$\frac{a^2(x-x_0)^2+b^2(y-y_0)^2}{(xy_0-x_0y)^2-(x-x_0)^2-(y-y_0)^2} = \frac{a^2(x-x_1)^2+b^2(y-y_1)^2}{(xy_1-x_1y)^2-(x-x_1)^2-(y-y_1)^2} \ldots (2),$$

represents the right line $L=0$ and a cubic curve.

1819. (Proposed by C. TAYLOR, M.A.)—From two fixed points on a given conic pairs of tangents are drawn to a variable confocal conic, and with the fixed points as foci a conic is described passing through any one of the four points of intersection. Show that its tangent or normal at that point passes through a fixed point.

Solution of the above Problems by PROFESSOR CAYLEY.

1. It is easy to see that drawing through the points A, C a circle, and through B, D a circle, such that the radii of the two circles are proportional to the lengths AC, BD, then that the required locus is that of the intersections of the two variable circles.

Take AC $=2l$, MO perpendicular to it at its middle point M, and $=p$; a, b the coordinates of M, and λ the inclination of p to the axis of x; then

coordinates of O are $\quad a+p\cos\lambda,\ b+p\sin\lambda$,
coordinates of A, C are $a\pm l\sin\lambda,\ b\mp l\cos\lambda$,

and hence the equation of a circle, centre O and passing through A, C, is

$$(x-a-p\cos\lambda)^2+(y-b-p\sin\lambda)^2=l^2+p^2;$$

or, what is the same thing,

$$(x-a)^2+(y-b)^2-l^2=2p\left[(x-a)\cos\lambda+(y-b)\sin\lambda\right].$$

If $2m$, q, c, d, μ refer in like manner to the points B, D, then the equation of a circle centre say Q, and passing through B, D, is

$$(x-c)^2+(y-d)^2-m^2=2q\left[(x-c)\cos\mu+(y-d)\sin\mu\right].$$

And the condition as to the radii is $l^2+p^2:m^2+q^2=l^2:m^2$, that is, $p^2:q^2=l^2:m^2$, or $p:q=\pm l:m$. And we thus have for the equation of the required locus

$$\frac{(x-a)^2+(y-b)^2-l^2}{(x-a)\cos\lambda+(y-b)\sin\lambda}=\pm\frac{l}{m}\,\frac{(x-c)^2+(y-d)^2-m^2}{(x-c)\cos\mu+(y-d)\sin\mu},$$

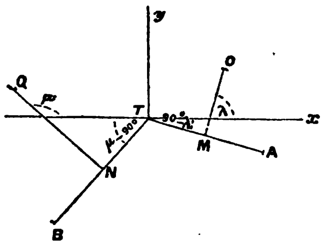
viz., the locus is composed of two cubics, which are at once seen to be *circular* cubics. One of these will however belong (at least for some positions of the four points) to the case of the subtended angles being *equal*, the other to that of the subtended angles being *supplementary*; and we may say that the required locus is a circular cubic.

2. If two of the points coincide, suppose C, D at T; then, taking T as the origin, we may write

$$a=\;\;l\sin\lambda,\quad b=-l\cos\lambda,$$
$$c=-m\sin\mu,\quad d=\;\;m\cos\mu,$$

and the equation becomes

$$\frac{x^2+y^2=2l\,(x\sin\lambda-y\cos\lambda)}{x\cos\lambda+y\sin\lambda}$$
$$=\pm\frac{l}{m}\,\frac{x^2+y^2+2m\,(x\sin\mu-y\cos\mu)}{x\cos\mu+y\sin\mu}$$



viz., this is

$$(x^2+y^2)\left[m\,(x\cos\mu+y\sin\mu)\mp l\,(x\cos\lambda+y\sin\lambda)\right]$$
$$-2lm\left\{(x\sin\lambda-y\cos\lambda)\,(x\cos\mu+y\sin\mu)\right.$$
$$\left.\pm(x\sin\mu-y\cos\mu)\,(x\cos\lambda+y\sin\lambda)\right\}=0.$$

Taking the lower signs, the term in $\{\;\}$ is $(x^2+y^2)\sin(\lambda-\mu)$, and the equation is

$$(x^2+y^2)\left\{m\,(x\cos\mu+y\sin\mu)+l\,(x\cos\lambda+y\sin\lambda)-2lm\sin(\lambda-\mu)\right\}=0,$$

viz., this is $x^2+y^2=0$, and a line which is readily seen to be the line AB; and in fact from any point whatever of this line the points A, T and the points B, T subtend *supplementary* angles.

Taking the upper signs, the equation is

$$(x^2+y^2)\left[m\,(x\cos\mu+y\sin\mu)-l\,(x\cos\lambda+y\sin\lambda)\right]$$
$$-2lm\left\{(x^2-y^2)\sin(\lambda+\mu)-xy\cos(\lambda+\mu)\right\}=0,$$

which is the locus for *equal* angles, a circular cubic as in the case of the four distinct points.

3. The Question is connected with Question 1819, which is given above. In fact, taking A, B for the fixed points on the given conic, and P for the intersection of any two of the tangents, if in the conic (foci A, B) which passes

through P, the tangent or normal at P passes through a fixed point T, then it is clear that at P the points A, T and B, T subtend equal angles, and consequently the locus of P should be a circular cubic as above. The theorem will therefore be proved if it be shown that the locus of P considered as the intersection of tangents from A, B to the variable confocal conic is in fact the foregoing circular cubic. I remark that the fixed point T is in fact the intersection of the tangents AT, BT to the given conic at the points A, B respectively.

4. Consider the points A, B, (which we may in the first instance take to be arbitrary points, but we shall afterwards suppose them to be situate on the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,) and from each of them draw a pair of tangents to the confocal conic $\frac{x^2}{a^2+h}+\frac{y^2}{b^2+h}=1$. Take (x_0, y_0) for the coordinates of A, and (x_1, y_1) for those of B; then the equation of the pair of tangents from A is

$$\left(\frac{x_0^2}{a^2+h}+\frac{y_0^2}{b^2+h}-1\right)\left(\frac{x^2}{a^2+h}+\frac{y^2}{b^2+h}-1\right)-\left(\frac{xx_0}{a^2+h}+\frac{yy_0}{b^2+h}-1\right)^2=0,$$

or, what is the same thing,

$$\frac{(xy_0-x_0y)^2}{(a^2+h)\,(b^2+h)}-\frac{(x-x_0)^2}{a^2+h}-\frac{(y-y_0)^2}{b^2+h}=0,$$

that is, $\quad (xy_0-x_0y)^2-(b^2+h)\,(x-x_0)^2-(a^2+h)\,(y-y_0)^2=0,$

or as this may also be written

$$(xy_0-x_0y)^2-b^2\,(x-x_0)^2-a^2\,(y-y_0)^2=h\left[(x-x_0)^2+(y-y_0)^2\right];$$

and similarly for the tangents from B we have

$$(xy_1-x_1y)^2-b^2\,(x-x_1)^2-a^2\,(y-y_1)^2=h\left[(x-x_1)^2+(y-y_1)^2\right];$$

in which equations the points (x_0, y_0), (x_1, y_1) are in fact any two points whatever.

5. Eliminating h, we have as the locus of the intersection of the tangents

$$\frac{(xy_0-x_0y)^2-b^2\,(x-x_0)^2-a^2\,(y-y_0)^2}{(x-x_0)^2+(y-y_0)^2}=\frac{(xy_1-x_1y)^2-b^2\,(x-x_1)^2-a^2\,(y-y_1)^2}{(x-x_1)^2+(y-y_1)^2},$$

which is apparently a quartic curve; but it is obvious *à priori* that the locus includes as part of itself the line AB which joins the two given points. In fact, there is in the series of confocal conics one conic which touches the line in question, and since for this conic one of the tangents from A and also one of the tangents from B is the line AB, we see that every point of the line AB belongs to the required locus. The locus is thus made up of the line in question and of the cubic curve.

6. To effect the reduction it will be convenient to write ax, by in the place of x, y, (ax_0, by_0, ax_1, by_1 in place of x_0, y_0, x_1, y_1,) and thus consider the equation under the form

$$\frac{a^2\,(x-x_0)^2+b^2\,(y-y_0)^2}{(xy_0-x_0y)^2-(x-x_0)^2-(y-y_0)^2}=\frac{a^2\,(x-x_1)^2+b^2\,(y-y_1)^2}{(xy_1-x_1y)^2-(x-x_1)^2-(y-y_1)^2}$$

It is to be shown that this equation represents the line $L=0$, and a cubic curve.

Writing for a moment $x_0 = x + \xi_0$, $y_0 = y + \eta_0$, and $x_1 = x + \xi$, $y_1 = y + \eta_1$, the equation becomes

$$\frac{a^2\xi_0^2 + b^2\eta_0^2}{(x\eta_0 - y\xi_0)^2 - \xi_0^2 - \eta_0^2} = \frac{a^2\xi_1^2 + b^2\eta_1^2}{(x\eta_1 - y\xi_1)^2 - \xi_1^2 - \eta_1^2},$$

and hence, multiplying out, the equation is at once seen to contain the factor $\xi_0\eta_1 - \xi_1\eta_0$ (which is in fact the determinant just mentioned), and when divested of this factor the equation is

$$a^2 \left[(x^2-1)(\xi_0\eta + \xi_1\eta_0) - 2xy\xi_0\xi_1\right]$$
$$-b^2 \left[(y^2-1)(\xi_0\eta_1 + \xi_1\eta_0) - 2xy\eta_0\eta_1\right].$$

Writing herein for ξ_0, η_0, ξ_1, η_1 their values, and consequently

$$\xi_0\xi_1 = x^2 - x(x_0 + x_1) + x_0x_1,$$
$$\eta_0\eta_1 = y^2 - y(y_0 + y_1) + y_0y_1,$$
$$\xi_0\eta_1 + \xi_1\eta_0 = 2xy - x(y_0 + y_1) - y(x_0 + x_1) + x_0y_1 + x_1y_0,$$

and arranging the terms, the equation is found to be

$$(a^2x^2 + b^2y^2)\left[-x(y_1 + y_0) - y(x_1 + x_0)\right] + (a^2x^2 + b^2y^2)(x_0y_1 + x_1y_0)$$
$$-2xy\left[a^2(1 + x_0x_1) - b^2(1 + y_0y_1)\right]$$
$$+ (a^2 - b^2)\left[x(y_1 + y_0) + y(x_1 + x_0) - (x_0y_1 + x_1y_0)\right] = 0,$$

which is the required cubic curve.

7. Restoring the original coordinates, or writing $\frac{x}{a}$, $\frac{y}{b}$, $\frac{x_0}{a}$, &c. in place of x, y, x_0, &c., we have

$$(x^2 + y^2)\left[-x(y_1 + y_0) + y(x_1 + x_0)\right]$$
$$+ (x^2 - y^2)(x_0y_1 + x_1y_0) - 2xy\,(a^2 - b^2 + x_0x_1 - y_0y_1)$$
$$+ (a^2 - b^2)\left[x(y_1 + y_0) + y(x_1 + x_0) - (x_0y_1 + x_1y_0)\right] = 0,$$

which is a circular cubic the locus of the intersections of the tangents from the arbitrary points (x_0, y_0), (x_1, y_1) to the series of confocal conics $\frac{x^2}{a^2+h} + \frac{y^2}{b^2+h} = 1$; the origin of the coordinates is at the centre of the conics.

8. Supposing that the points (x_0, y_0), (x_1, y_1) are on the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and that we have consequently $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$, $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$, the equations of the tangents at these points respectively are

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1, \quad \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1;$$

and hence, writing for shortness $\alpha = y_0 - y_1$, $\beta = x_1 - x_0$, $\gamma = x_0y_1 - x_1y_0$, we find $x = -\frac{a^2\alpha}{\gamma}$, $y = -\frac{b^2\beta}{\gamma}$ as the coordinates of the point of intersection T, of the two tangents; and in order to transform to this point as origin, we must in place of x, y write $x - \frac{a^2\alpha}{\gamma}$, $y - \frac{b^2\beta}{\gamma}$ respectively. Or what is more convenient, we may in the equation at the end of (6), in which it is to be now assumed that $x_0^2 + y_0^2 = 1$, $x_1^2 + y_1^2 = 1$, write $x - \frac{\alpha}{\ }$, $y - \frac{\beta}{\gamma}$ for x, y, and

then restore the original coordinates by writing $\frac{x}{a}$, $\frac{y}{b}$, $\frac{x_0}{a}$, &c., for x, y, x_0, &c., and $\frac{\alpha}{b}$, $\frac{\beta}{a}$, $\frac{\gamma}{ab}$ for α, β, γ, these quantities throughout signifying $\alpha = y_0 - y_1$, $\beta = x_1 - x_0$, $\gamma = x_0 y_1 - x_1 y_0$. I however obtained the equation referred to the point T as origin by a different process, as follows:—

9. Starting from the equation at the commencement of (5), I found that the points (x_0, y_0), (x_1, y_1) being on the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the equation could be transformed into the form

$$\frac{\left(\frac{xx_0}{a^2} + \frac{yy_0}{b^2} - 1\right)^2}{(x-x_0)^2 + (y-y_0)^2} = \frac{\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2}{(x-x_1)^2 + (y-y_1)^2},$$

an equation which (not, as the original one, for all values of (x_0, y_0), (x_1, y_1), but) for values of (x_0, y_0), (x_1, y_1), such that $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$, $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$, breaks up into the line AB and a cubic curve.

10. To simplify the transformation, write as before ax, by, ax_0, &c., for x, y, x_0, &c. We have thus to consider the equation

$$\frac{a^2(x-x_0)^2 + b^2(y-y_0)^2}{(xx_0 + yy_0 - 1)^2} = \frac{a^2(x-x_1)^2 + b^2(y-y_1)^2}{(xx_1 + yy_1 - 1)^2},$$

where $x_0^2 + y_0^2 = 1$, $x_1^2 + y_1^2 = 1$, and which equation, I say, breaks up into the line $L = 0$, and into a cubic.

Write for shortness $\alpha = y_0 - y_1$, $\beta = x_1 - x_0$, $\gamma = x_0 y_1 - x_1 y_0$, so that the equation of the last mentioned line is $\alpha x + \beta y + \gamma = 0$. Then it may be verified that, in virtue of the relations between (x_0, y_0), (x_1, y_1), we have identically

$$(x-x_0)(xx_1 + yy_1 - 1) + (x-x_1)(xx_0 + yy_0 - 1) = (\alpha x + \beta y + \gamma)\frac{x_0 + x_1}{\alpha\gamma}(\gamma x + \alpha),$$

$$(x-x_0)(xx_1 + yy_1 - 1) - (x-x_1)(xx_0 + yy_0 - 1) = \beta x^2 - \alpha xy - \gamma y - \beta;$$

and, similarly,

$$(y-y_0)(xx_1 + yy_1 - 1) + (y-y_1)(xx_0 + yy_0 - 1) = (\alpha x + \beta y + \gamma)\frac{y_0 + y_1}{\beta\gamma}(\gamma y + \beta).$$

$$(y-y_0)(xx_1 + yy_1 - 1) - (y-y_1)(xx_0 + yy_0 - 1) = \beta xy - \alpha y^2 + \gamma x + \alpha.$$

11. The equation in question may be written $a^2 P + b^2 Q = 0$, where

$$P = (x-x_0)^2 (xx_1 + yy_1 - 1)^2 - (x-x_1)^2 (xx_0 + yy_0 - 1)^2,$$
$$Q = (y-y_0)^2 (xx_1 + yy_1 - 1)^2 - (y-y_1)^2 (xx_0 + yy_0 - 1)^2,$$

values which are given by means of the formulæ just obtained; there is a common factor $\alpha x + \beta y + \gamma$ which is to be thrown out; and we have also, as is at once verified, $\frac{y_0 + y_1}{\beta} = \frac{x_0 + x_1}{\alpha}$, so that these equal factors may be thrown out. We thus obtain the cubic equation

$$a^2(\gamma x + \alpha)(\beta x^2 - \alpha xy - \gamma y - \beta) + b^2(\gamma y + \beta)(\beta xy - \alpha y^2 + \gamma x + \alpha) = 0.$$

This is simplified by writing $x-\frac{\alpha}{\gamma}$ for x, $y-\frac{\beta}{\gamma}$ for y. It thus becomes

$$a^2x\left[(\gamma x-\alpha)(\beta x-\alpha y)-\gamma^2 y\right]+b^2y\left[(\gamma y-\beta)(\beta x-\alpha y)+\gamma^2 x\right]=0;$$

or, what is the same thing,

$$a^2x\left[\gamma x(\beta x-\alpha y)-\alpha\beta x+(\alpha^2-\gamma^2)y\right]+b^2y\left[\gamma y(\beta x-\alpha y)-(\beta^2-\gamma^2)x+\alpha\beta y\right]=0;$$

that is, $\gamma(a^2x^2+b^2y^2)(\beta x-\alpha y)+a^2\left[-\alpha\beta x^2+(\alpha^2-\gamma^2)xy\right]$
$+b^2\left[-(\beta^2-\gamma^2)xy+\alpha\beta y^2\right]=0.$

12. Restoring $\frac{x}{a}, \frac{x_0}{a}, \frac{x_1}{a}$ for x, x_0, x_1, and $\frac{y}{a}, \frac{y_0}{a}, \frac{y_1}{a}$ for y, y_0, y_1; writing, consequently, $\frac{\alpha}{b}, \frac{\beta}{a}, \frac{\gamma}{ab}$ in place of α, β, γ, if α, β, γ are still used to denote $\alpha=y_0-y_1$, $\beta=x_1-x_0$, $\gamma=x_0y_1-x_1y_0$, the equation becomes

$$\gamma(x^2+y^2)\left[b^2\beta x-a^2\alpha y\right]+a^2\left[-b^2\alpha\beta x^2+(a^2\alpha^2-\gamma^2)xy\right]$$
$$+b^2\left[-(b^2\beta^2-\gamma^2)xy+a^2\alpha\beta y^2\right]=0,$$

where now, as originally, $\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$, $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$; viz., this is the equation, referred to the point T as origin, of the locus of the point P considered as the intersection of tangents from A, B to the variable confocal conic; and it is consequently the equation which would be obtained as indicated in (8). The locus is thus a circular cubic; the equation is identical in form with that obtained (2) for the locus of the point at which A, T and B, T subtend equal angles, and the complete identification of the two equations may be effected without difficulty.

13. I may remark that M. Chasles has given (Comptes Rendus, tom. 58, February, 1864) the theorem that the locus of the intersections of the tangents drawn from a fixed conic to the conics of a system (μ, ν) is a curve of the order 3ν. The confocal conics, *quâ* conics touching four fixed lines, are a system $(0, 1)$; hence, taking for the fixed conic the two points A, B, we have, as a very particular case, the foregoing theorem, that the locus of the intersections of the tangents drawn from two fixed points to a variable confocal conic is a cubic curve.

2778. (Proposed by Professor SYLVESTER.)—Suppose μ any quantity fractional or integral, and write

$$c_0=a_0;\quad c_1=\mu a_1,\quad c_2=\frac{\mu(\mu+1)}{2}a_2,\quad c_3=\frac{\mu(\mu+1)(\mu+2)}{2.3}a_3,\quad \ldots;$$

also let n be a positive integer such that μ and $\mu+n$ have not different signs; then it has been proved elsewhere, and may be assumed to be true, that the number of changes of sign in the series

$$a_0^2,\quad a_1^2-a_0a_2,\quad a_2^2-a_1a_3,\quad \ldots$$

is *an inferior limit* to the number of imaginary roots in the equation

$$c_0x^n+c_1x^{n-1}+c_2x^{n-2}+\ldots=0.$$

1. Required to prove that the above theorem remains true when we write

$$c_0 = a^0,\ c_1 = \frac{\mu}{\nu}a_1,\ c_2 = \frac{\mu(\mu+1)}{\nu(\nu+1)}a_2,\ c_3 = \frac{\mu(\mu+1)(\mu+2)}{\nu(\nu+1)(\nu+2)}a_3,\ \ldots.,$$

provided ν is a positive integer, and $\mu-\nu+1$, $\mu+\nu$ do not differ in sign.

2. Ascertain whether ν being an integer is essential to the truth of the extended theorem.

Solution by the PROPOSER.

Let $\quad fx = c_0x^n + c_1x^{n-1} + \ldots.,\qquad Fx = x^{n+1}\phi x + fx,$

$$\phi x = \epsilon x^{\nu-1} + \mu\theta\epsilon^2x^{\nu-2} + \frac{\mu(\mu+1)}{1.2}\epsilon^3x^{\nu-3} + \frac{\mu(\mu+1)(\mu+2)}{1.2.3}\theta\epsilon^4x^{\nu-4} + \ldots.,$$

the alternately appearing and disappearing factor θ being any quantity different from unity, ϵ an infinitesimal of like sign with c_0, ν any positive integer, and μ any quantity (fractional or integer) such that μ and $\mu+n+\nu$ do not differ in sign.

Let $\ c_0 = \dfrac{\mu(\mu+1)\ldots.(\mu+\nu-1)}{1.2\ldots.\nu}a_0;\quad c_1 = \dfrac{\mu(\mu+1)\ldots.(\mu+\nu)}{1.2\ldots.(\nu+1)}a_1;\ \ldots.$

Then, by the Postulate of the question, the number of imaginary roots in Fx is not less than the sum of the changes of sign in the two series

$$\epsilon^2;\ (\theta^2-1)\epsilon^4;\ (1-\theta^2)\epsilon^6;\ \ldots.;\ (\ldots.);\ a^2;$$

$$a^2;\ a_1^2-a_0a_2;\ \ldots.;\ a_n^2.$$

In each of these series, the term a^2, properly speaking, has attached to itself another term, but which may be rejected as of only infinitesimal magnitude. In the penultimate term of the upper series, the parenthesis left vacant (similarly neglecting an infinitesimal term of higher order than the one retained) will be $-\epsilon^{\nu-1}a_0$ or $-\theta\epsilon^{\nu-1}a_0$, according as ν is even or odd. Thus the number of changes of sign in this series is ν if ν is even, and $\nu-1$ if ν is odd.

Again, the roots of Fx consist of a group of infinitely great and a group of finite magnitudes. The former are the roots of the equation $x\phi x + c_0 = 0$, or, which is ultimately the same thing, of $\epsilon x^\nu + c_0 = 0$; the latter of $fx = 0$. The number of imaginary roots of the infinite group is therefore ν or $\nu-1$ according as ν is even or odd; *i. e.*, is the same as the number of changes of sign in the upper series. Hence the number of imaginary roots belonging to $fx=0$ cannot be less than the number of sign-changes in the lower series; but this last number will not be affected if we multiply each quantity, $a_0, a_1, a_2, \ldots.$, by the common factor $\dfrac{\mu(\mu+1)\ldots.(\mu+\nu-1)}{1.2\ldots.\mu}$, or, which is the same thing, if we write

$$c_0 = a_0;\quad c_1 = \frac{\mu+\nu}{\nu+1}a_1;\quad c_2 = \frac{(\mu+\nu)(\mu+\nu-1)}{(\nu+1)(\nu+2)}a_2;\ \ldots.$$

Let now $\nu+1 = \nu'$, $\mu+\nu = \mu'$, so that

$$\mu = \mu'-\nu = \mu'-\nu'+1,\quad \text{and}\quad \mu+n+\nu = \mu'+n.$$

Then, suppressing the accents, we see that, ν being any positive integer, and μ not intermediate between $\nu-1$ and $-n$, if

$$c_0 = a_0;\quad c_1 = \frac{\mu}{\nu}a_1;\quad c_2 = \frac{\mu(\mu+1)}{\nu(\nu+1)}a_2;\ \ldots.$$

the number of imaginary roots in fx is not less than that of sign-changes in

$$a_0^2;\ a_1^2-a_0a_2;\ a_2^2-a_1a_3;\ \ldots.$$

When $\nu=1$ and $\mu=-n$, we fall back again upon Newton's rule. One feels strongly inclined (on more than one ground) to suspect that ν need not be integer, but I do not see my way to a proof.

As a corollary, it may be noticed that the equation

$$x^n+\frac{\mu}{\nu}x^{n-1}+\frac{\mu(\mu+1)}{\nu(\nu+1)}x^{n-2}+\ldots,$$

or more generally, the real function (where θ^2 does not exceed unity)

$$\left(x^n+\frac{\mu(\mu+1)}{\nu(\nu+1)}x^{n-2}+\frac{\mu(\mu-1)(\mu+2)(\mu+3)}{\nu(\nu+1)(\nu+2)(\nu+3)}x^{n-4}+\ldots\right)$$
$$+\theta\left(\frac{\mu}{\nu}x^{n-1}+\frac{\mu(\mu+1)(\mu+2)}{\nu(\nu+1)(\nu+2)}x^{n-3}+\ldots\right),$$

when ν is a positive integer, and $\mu-\nu+1$, $\mu+n$ have the same sign (*i. e.* when μ is not intermediate between $\nu-1$ and $-n$), cannot have more than one real root.

N.B.—It is worthy of notice, that if we make fx become the *middle part* of a function of x having for its roots a group of infinitely greats, a group of infinitely smalls, and the roots of fx itself, we may, by a method of demonstration similar to that here employed, pass direct from Newton's rule (as given by Newton) to the final extension of it contained in the question above, but with the difference that, so far as the force of such demonstration takes effect, μ as well as ν will have to be integral, a restriction we know to be superfluous. This is one of the reasons for doubting the necessity of ν being an integer.

For the proof of the Postulate, see the Proceedings of the London Mathematical Society, No. II. The above demonstration would have been just as good if only the first powers of the infinitesimal ϵ, instead of the ascending scale, had been used in ϕx throughout; but I have not thought it worth while to disturb the text by making this simplification, which the reader can do for himself.

To find the number of permutations of n *things taken* r *together.*

By C. R. Rippin, M.A.

Let $P(n, r)$ denote the number. Now, if n things be arranged in all possible ways r at a time, each will stand first as often as the remaining $n-1$ can be arranged $r-1$ at a time, that is, $P(n-1, r-1)$ times; but the whole number of permutations is secured by making each thing stand first as often as possible; and therefore, since there are n things, we must have

$$P(n, r) = n\,.\,P(n-1, r-1).$$

Similarly $P(n-1, r-1) = (n-1)\,.\,P(n-2, r-2)$,

&c. = &c.

$$P(n-r+2, 2) = (n-r+2)\,.\,P(n-r+1, 1) = (n-r+2)(n-r+1.)$$

Multiply and simplify, then we obtain

$$P(n, r) = n(n-1)(n-2)\ldots(n-r+2)(n-r+1).$$

2798. (Proposed by J. J. WALKER, M.A.)—To find the condition that a given point (x, y, z) may be exterior to the given conic

$$S \equiv (A, B, C, F, G, H \between x, y, z)^2 = 0.$$

Solution by F. D. THOMSON, M.A.

If a point be without a conic, its polar cuts the conic in two real points. Now let $\lambda x + \mu y + \nu z = 0$ be the equation to polar of x, y, z.

Then, for the points of intersection of the polar with the conic, we have the equation

$$(A, B, C, F, G, H \between -(\mu y + \nu z), \lambda y, \lambda z)^2 = 0.$$

Expanding this as a quadratic in $\dfrac{y}{z}$, and writing down the condition for *real* roots, we have

$$(F^2 - BC, G^2 - CA, H^2 - AB, AF - GH, BG - FH, CH - FG \between \lambda, \mu, \nu)^2 > 0;$$

or, since $\qquad \lambda : \mu : \nu = \dfrac{dS}{dx} : \dfrac{dS}{dy} : \dfrac{dS}{dz},$

$$\left(A', B', C', F', G', H' \between \frac{dS}{dx}, \frac{dS}{dy}, \frac{dS}{dz}\right)^2 > 0 \ldots\ldots\ldots\ldots (1),$$

where A', B', &c. are the *inverse* coefficients.

The condition (1) reduces to

$$(AF^2 + BG^2 + CH^2 - ABC - 2FGH)\, S > 0, \quad \text{or} \quad \Delta S > 0 \ldots\ldots (2).$$

In the particular case where $y = z = 0$, $S = Ax^2$ and is positive; therefore the condition becomes $\Delta > 0$.

1912. (Proposed by Professor MANNHEIM.)—If a and b be the points of contact, with a curve of the third class, of a double tangent; and if this tangent be intersected in m, n, p by the three tangents to the curve which can be drawn from any point in the plane, then $\dfrac{am \,.\, an \,.\, ap}{bm \,.\, bn \,.\, bp} = \dfrac{\rho_a}{\rho_b}$, where ρ_a, ρ_b are the radii of curvature at the points a and b.

Solution by the Rev. J. WOLSTENHOLME, M.A.

Take (x, y, z) for tangential coordinates, and (α, β, γ) for areal coordinates. The tangential equation of a curve of the third class, to which $\alpha = 0$ is a double tangent at the points where it meets $\beta = 0$, $\gamma = 0$, is of the form

$$xyz + Ay^3 + By^2z + Cyz^2 + Dz^3 = 0 \ldots\ldots\ldots\ldots\ldots (1).$$

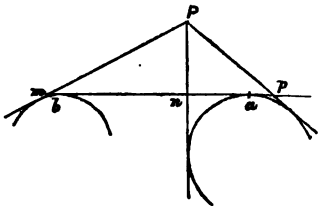
For the tangents through an arbitrary point (p, q, r) we shall have

$$px + qy + rz = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

which, combined with (1), leads to the cubic

$$p\,(Ay^3+By^2z+Cyz^2+Dz^3)-yz\,(qy+rz)=0\,;$$

so that $\dfrac{y_1y_2y_3}{z_1z_2z_3}=\dfrac{D}{A}$ and is independent of (p,q,r). But the tangent $x_1\alpha+y_1\beta+z_1\gamma=0$ meets the straight line $\alpha=0$ in the point where $y_1\beta+z_1\gamma=0$, or $\dfrac{am}{bm}=\dfrac{y_1}{z_1}$; whence $\dfrac{am\,.\,an\,.\,ap}{bm\,.\,bn\,.\,bp}$ is constant.



To investigate this constant, take a point P near a, from which draw three tangents, and let $\angle\,Pmn=\delta\phi$, $Ppm=\delta\phi'$. Then the constant required is the limit of $\dfrac{am\,.\,an\,.\,ap}{bm\,.\,bn\,.\,bp}$ when m moves up to b, and n and p to a,

$$=\text{limit of }\frac{ab\,.\,an\,.\,ap}{\tfrac{1}{2}\rho_b\,\delta\phi\,.\,ab\,.\,ab}$$

$$=\frac{2}{ab\,.\,\rho_b}\ \text{limit of }\frac{an\,.\,ap}{d\phi}\,;\quad\text{and}\quad\frac{Pp}{\delta\phi}=\frac{Pm}{\delta\phi'}=\frac{ab}{\delta\phi'}$$

$$=\frac{2}{\rho_b}\ \text{limit of }\frac{an\,.\,ap}{Pp\,.\,\delta\phi'}\,;$$

and making P approach its limit in such a direction that $an=ap$, we shall therefore have ultimately $\dfrac{Pp}{an}=1$, and $ap=\tfrac{1}{2}\rho_a\,\delta\phi'=\dfrac{\rho_a}{\rho_b}$.

[A solution by Professor CREMONA has been given on p. 88 of Vol. VII. of the *Reprint*.]

2740. (Proposed by W. S. BURNSIDE, M.A.)—Prove that the envelope of a line given by an equation of the form

$$L\cos 2\phi+M\sin 2\phi+P\cos\phi+Q\sin\phi+R=0$$

may be obtained as the discriminant of a cubic equation, and form this equation.

Solution by the PROPOSER.

The above equation is plainly reducible to the form

$$(a\,b\,c\,f\,g\,h)\,(\cos\phi,\sin\phi,1)^2=0,$$

which further is equivalent to $(a\,b\,c\,f\,g\,h)\,(\xi,\eta,\zeta)^2=0$ or $U=0$, with $\xi^2+\eta^2-\zeta^2=0$ or $V=0$; whence, if we find the condition that the conics U and V touch (considering $a\,b\,c\,f\,g\,h$ as constants), we shall have the equation of the envelope required. And this is most easily done by forming the discriminant of $U+\lambda V=0$, *sc.* $(ABCD)\,(1,\lambda)^3$, and then equating to zero the discriminant of this cubic.

2636. (Proposed by Professor SYLVESTER.)—
1. If A, B be two confocal conics, and A′ be a third conic having double contact with A, show that there exists a fourth conic B′ having double contact with B and confocal with A′.

2. If A, B be two confocal quadrics, and A′ be a third quadric having continuous contact with A, show that there exists a fourth quadric B′ having continuous contact with B and confocal with A′.

Solution by the Rev. J. WOLSTENHOLME, M.A.

1. Generalized by projection, this amounts to: Given two conics A, B inscribed in the same quadrilateral (of which OO′ is a diagonal); there can be described two conics A′, B′ having double contact with A, B respectively, and such that the tangents to the two from O, O′ are the same; or (to express it more clearly) two of whose common tangents intersect in O and the other two in O′. Reciprocating this, we have two conics a, b passing through four points lying on the straight lines o, o', and the theorem to be proved is that two conics a', b' can be formed having double contact with a, b respectively and meeting the straight lines o, o' in the same points. The simplest case to which this general theorem can be reduced by projection appears to be: Given two conics A, B similar and similarly situated, two circles can be drawn having double contact with A, B respectively, and meeting their common chord in the same two points. This is not at all hard to prove; but as I have not seen how to get a neat solution, I refrain.

2369. (Proposed by H. R. GREER, B.A.)—Given a cubic curve K, and a point on it p; through P is drawn any transversal meeting K again in m and n, and on it is taken a point x such that the anharmonic ratio $(pxmn)$ shall be equal to a given quantity. Prove that the locus of x is a quartic curve with a point of osculation at p, touching K at p (counting for four points of intersection) and in four other points.

Solution by JAMES DALE.

Referring K to rectangular axes passing through p, its equation is of the form $$Ax^3+Bx^2y+Cxy^2+Dy^3+Ex^2+Fxy+Gy^2+Hx+Ky=0,$$
which, being transformed to polar coordinates, becomes
$$r^2(A\cos^3\theta+B\cos^2\theta\sin\theta+C\cos\theta\sin^2\theta+D\sin^3\theta)$$
$$+r(E\cos^2\theta+F\cos\theta\sin\theta+G\sin^2\theta)+H\cos\theta+K\sin\theta=0.$$

Let now x be a point on any radius vector cutting the curve in m and n, so that the anharmonic ratio $=l:m$; then, putting $px=r$, $pm=r_1$, $pn=r_2$, we get $$r\left\{(l-m)(r_1+r_2)+(l+m)(r_1-r_2)\right\}=2(l-m)r_1r_2.$$
Substituting the values (r_1+r_2), (r_1-r_2), r_2r_2, obtained from the equation to the curve, and returning to x and y coordinates, we get

$$(l-m)^2\left\{Ex^2+Fxy+Gy^2+2(Hx+Ky)\right\}^2$$
$$=(l+m)^2\left\{Ex^2+Fxy+Gy^2-4(Hx+Ky)(Ax^3+Bx^2y+Cxy^2+Dy^3)\right\}$$

as the equation to the locus of x, which is evidently of the fourth degree.

Let $Ex^2+Fxy+Gy^2+2(Hx+Ky)=0$, which is the equation of the polar conic of p with respect to K, be represented by P_2; and let $Hx+Ky=0$, which is the common tangent at p_1 to P and K, be represented by T_1, and the cubic by K_3; then the locus of x may be written

$$lmP_2^2=2(l+m)^2T_1.K_3;$$

and it may be shown that this curve touches K_3 in the points common to T_1 and P_2, that is, in two coincident points at the origin, T_1 being the common tangent to the three curves, and in four other points.

2494. (Proposed by Professor EVERETT.)—On each side of a hexagon as base, a triangle is described by producing the two adjacent sides to meet, and a second hexagon is formed by joining the vertices of these triangles in order. Show that if either of these two hexagons can be inscribed or circumscribed to a conic, the other can be circumscribed or inscribed to a conic.

2513. (Proposed by Professor EVERETT.)—Let A, B, C, D, E, F be six points. Let BF, CD meet in a; AB, CE in b; AB, CD in c; AF, CE in d; BF, DE in e; and AF, DE in f. Show that, if either of the hexagons ABCDEF, $abcdef$ can be inscribed or circumscribed to a conic, the other can be circumscribed or inscribed to a conic.

2539. (Proposed by Professor EVERETT.)—If three conics cut one another so that every two of them have a common chord which is also a chord of a fourth conic, the other three chords of intersection of the three conics meet in a point.

Solution by the PROPOSER.

THEOREM I. [Quest. 2539.] Let S be the fourth conic (*i. e.*, let its equation be $S=0$), and α, β, γ the three chords common to it and to the others two by two; then $S-l\beta\gamma$, $S-m\gamma\alpha$, $S-n\alpha\beta$ will be the three conics. By subtraction between the two last, we have $m\gamma\alpha-n\alpha\beta$ or $\alpha\left(\frac{\beta}{m}-\frac{\gamma}{n}\right)$, denoting a pair of lines which must be chords of intersection of two of the three conics; and it is obvious that $\frac{\beta}{m}-\frac{\gamma}{n}$, $\frac{\gamma}{n}-\frac{\alpha}{l}$, $\frac{\alpha}{l}-\frac{\beta}{m}$ meet in a point.

THEOREM II. If the three conics break up into pairs of lines, the foregoing theorem has in general, for any given values of α, β, γ, eight distinct applications, that is to say, there will be eight points in each of which three lines meet.

There are two values of l which make $S-l\beta\gamma$ break up into a pair of lines.

For instance, if AC and BD are β and γ, $S-l\beta\gamma$ may denote either AB and DC or AD and BC. We have thus six pairs of lines which we may denote by

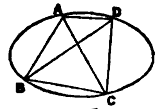


FIG 1.

$$S-l\beta\gamma,\qquad S-m\gamma\alpha,\qquad S-n\alpha\beta,$$
$$S-l'\beta\gamma,\qquad S-m'\gamma\alpha,\qquad S-n'\alpha\beta,$$

and eight points of meeting, viz.,

$$\frac{\alpha}{l}=\frac{\beta}{m}=\frac{\gamma}{n};\quad \frac{\alpha}{l'}=\frac{\beta}{m'}=\frac{\gamma}{n'};\quad \frac{\alpha}{l}=\frac{\beta}{m}=\frac{\gamma}{n'},\quad \text{\&c.,}$$

which we may denote by lmn, $l'm'n'$, $l'mn$, $lm'n$, lmn', $lm'n'$, $l'mn'$, $l'm'n$; and the pairs of lines may in like manner be denoted by l, l', m, m', n, n'. Each of the lines passes through two of the points.

Let A, B, C, D, E, F be any six points on a conic, and let α, β, γ denote AD, BE, CF respectively; then

l is BC and EF, m is CD and FA, n is DE and AB;
l' is BF and CE, m' is FD and AC, n' is BD and EA.

$\frac{\beta}{m}-\frac{\gamma}{n}$ will denote the line joining the intersection of AB and CD to that of FA and DE, which we shall indicate by the notation "AB, CD to FA, DE." The eight sets of lines are

AB, CD to FA, DE;	BC, DE to AB, EF;	CD, EF to BC, FA	meeting in	$l\,m\,n'$
AC, BD to DF, EA;	BD, CE to EA, FB;	CE, DF to FB, AC		$l'm'n$
AB, CD to FA, DE;	AB, CE to DE, BF;	AF, CE to CD, BF		$l'm\,n'$
BC, DE to AB, EF;	BC, DF to EF, CA;	BA, DF to DE, CA		$l\,m'n$
CD, EF to BC, FA;	CD, EA to FA, DB;	CB, EA to EF, DB		$l\,m\,n'$
AC, BD to DF, EA;	EF, BD to BC, EA;	EF, AC to BC, DF		$l\,m'n$
BD, CE to EA, FB;	FA, CE to CD, FB;	FA, BD to CD, EA		$l'm\,n'$
CE, DF to FB, AC;	AB, DF to DE, AC;	AB, CE to DE, FB		$l'm'n$

The figures of the first two cases are subjoined. In both of them, as here lettered, the lines meeting in a point are GL, HM, and KN. It will be observed that the hexagon GHKLMN is formed from ABCDEF, in Fig. 1, in the same way as ABCDEF is formed from GHKLMN, in Fig. 2. A similar relation connects $l'mn$ with $lm'n'$, $lm'n$ with $l'mn'$, and lmn' with $l'm'n$. The following figure, in which the conic may be supposed to pass either through A, B, C, D, E, F, or through a, b, c, d, e, f,

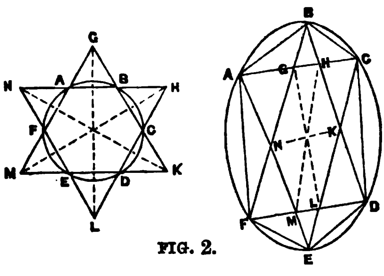


FIG. 2.

applies to the two cases $l'mn$ and $lm'n'$, the capital letters being applicable to the former and the small letters to the latter. In the former case, ad, be, cf will meet in a point; in the latter, AD, BE, CF. We have thus proved that, if either of the two hexagons (mentioned in Questions 2494, 2513) can be inscribed in a conic, the diagonals of the other meet in a point; whence it follows, by the converse of Brianchon's theorem, that this other hexagon can be circumscribed about a conic.

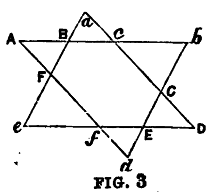


FIG. 3

Note.—Each of the eight sets of lines is connected with a Pascal's line by the relation that, if two triangles be such that the lines joining corresponding vertices meet in a point, the three intersections of corresponding sides will be in the same straight line. In Fig. 2 the two triangles are GKM, LNH. It will be found that the line $\frac{\beta}{m}=\frac{\gamma}{n}$ is the Pascal ABEDCF, and that $\frac{\beta}{m'}=\frac{\gamma}{n'}$ is the Pascal DBEACF.

2101. (Proposed by M. DARBOUX.)—Trouver les conditions nécessaires et suffisantes pour que les quatre racines d'une équation du quatrième degré forment un quadrilatère inscriptible. Trouver la surface et le rayon de ce quadrilatère.

Solution by J. J. WALKER, M.A.

If (a, b, c, d) be the four real and positive roots of a biquadratic $x^4-px^3+qx^2-rx+s=0$, it is evident that the condition of its being possible to construct a quadrilateral figure the four sides of which shall be represented in length by these roots is, simply, that the greatest of these values should be less than the sum of the other three; *i. e.*, that the expression

$$u=(a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d)$$

should be positive. In terms of the coefficients of the biquadratic

$$\begin{aligned} u &\equiv 2\Sigma a^2b^2-\Sigma a^4+8abcd \equiv 4\Sigma a^2b^2-(\Sigma a^2)^2+8abcd \\ &= 4(q^2-2pr+2s)-(p^2-2q)^2+8s = p^2(4q-p^2)+8(2s-pr). \end{aligned}$$

This condition being fulfilled, let a, b be two adjacent sides of the figure, θ the angle between them, and δ the length of the diagonal joining their ends; and let θ' be the angle between the remaining sides c, d. Then, if the figure is to be inscribable in a circle of radius R, we have

$$\cos\theta+\cos\theta'=0, \quad \text{whence} \quad \delta^2=\frac{(a^2+b^2)cd+(c^2+d^2)ab}{ab+cd}; \text{ and}$$

$$\cos\theta=\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}, \text{ whence } \sin^2\theta=\frac{4(ab+cd)^2-\left\{a^2+b^2-(c^2+d^2)\right\}^2}{4(ab+cd)^2}$$

$$=\frac{2\Sigma a^2b^2-\Sigma a^4+8abcd}{4(ab+cd)^2}=\frac{p^2(4q-p^2)+8(2s-pr)}{4(ab+cd)^2}.$$

The area of the quadrilateral $=\frac{1}{2}(ab+cd)\sin\theta$

$$=\tfrac{1}{4}\left\{p^2(4q-p^2)+8(2s-pr)\right\}\ldots\ldots\ldots\ldots(2).$$

$$\text{Again} \quad R^2=\frac{\delta^2}{4\sin^2\theta}=\frac{\left\{(a^2+b^2)cd+(c^2+d^2)ab\right\}(ab+cd)}{u}.$$

This numerator is a symmetric function of a, b, c, d, being equal to $\Sigma a^3bcd+\Sigma a^2b^2c^2=(p^2-2q)s+r^2-2qs=(p^2-4q)s+r^2$, which is, plainly, essentially positive. Hence we obtain

$$R^2=\frac{(p^2-4q)s+r^2}{p^2(4q-p^2)+8(2s-pr)}\ldots\ldots\ldots\ldots(3).$$

In the case of a convex quadrilateral, then, the only condition to be fulfilled by the real and positive roots of the biquadratic is $u=0$; and such a figure is always inscribable in a real circle by giving its angles the values determined by $\sin\theta = \dfrac{\sqrt{u}}{2(ab+cd)}$.

If the quadrilateral is to be *skew* or *twisted*,

$$\delta^2 = \frac{(a^2+b^2)\,cd-(c^2+d^2)\,ab}{cd-ab};$$

and that this should be positive, $\left\{(a^2+b^2)\,cd-(c^2+d^2)\,ab\right\}(cd-ab)$ must be positive; or $\Sigma a^2b^2c^2-\Sigma a^3bcd$, *i. e.*, $r^2-2qs-(p^2-2q)\,s$, or r^2-p^2s, must be positive. If this condition and $u=0$ are satisfied, it will be found that

$$\sin^2\theta = \frac{4(ab-cd)^2-(a^2+b^2-c^2-d^2)^2}{4(ab-cd)^2} = \frac{p\,(4pq-p^3-8r)}{4\,(ab-cd)^2};$$

$$\text{Area} = \tfrac{1}{4}p\,(4pq-p^3-8r); \text{ and } R^2 = \frac{r^2-p^2s}{p\,(4pq-p^3-8r)}.$$

2616. (Proposed by C. W. MERRIFIELD, F.R.S.)—Let two intersecting tetrahedra have all their edges bisected by the same system of Cartesian axes, each axis through two opposite edges of each tetrahedron; then the solid about the origin has the origin for its centre of figure.

Solution by the PROPOSER.

The origin is of course the centre of figure of each tetrahedron separately. Moreover each tetrahedron is completely defined by the angles between the axes, and the intercepts of the edges, except that it may be inverted.

Calling the intercepts a, b, c, and writing the equation $\dfrac{x}{p}-\dfrac{y}{q}+\dfrac{z}{r}=1$ in the abbreviated form $p\,\bar{q}\,r$, the equations of the sides of one tetrahedron are

$$a\,b\,c,\quad a\,\bar{b}\,\bar{c},\quad \bar{a}\,b\,\bar{c},\quad \bar{a}\,\bar{b}\,c,$$

while those of the inverse tetrahedron are

$$\bar{a}\,\bar{b}\,\bar{c},\quad \bar{a}\,b\,c,\quad a\,\bar{b}\,c,\quad a\,b\,\bar{c}.$$

The two systems of planes form an octohedron.

I begin with the clearest case of intersection, in which the corners of the inner tetrahedron simply project through the sides of the outer one, without crossing its edges.

Let the planes of the tetrahedra be represented by the following groups of planes, opposite planes (not necessarily parallel) written under one another,

$$\begin{array}{cccc} a\,b\,c, & a\,\bar{b}\,\bar{c}, & \bar{a}\,b\,\bar{c}, & a\,b\,\bar{c}, \\ \bar{l}\,\bar{m}\,\bar{n}, & \bar{l}\,m\,n, & l\,\bar{m}\,n, & l\,m\,\bar{n}. \end{array}$$

The secondary pyramid made by the plane $\bar{l}\,\bar{m}\,\bar{n}$ with the three planes $a\,\bar{b}\,\bar{c}$, $\bar{a}\,b\,\bar{c}$, $\bar{a}\,\bar{b}\,c$ has its four points $(x_0 y_0 z_0)$, &c. determined by the four systems of equations

$$\left.\begin{matrix} a\,\bar{b}\,\bar{c} \\ \bar{a}\,b\,\bar{c} \\ \bar{a}\,\bar{b}\,c \end{matrix}\right\} \qquad \left.\begin{matrix} \bar{l}\,\bar{m}\,\bar{n} \\ \bar{a}\,b\,\bar{c} \\ \bar{a}\,\bar{b}\,c \end{matrix}\right\} \qquad \left.\begin{matrix} \bar{l}\,\bar{m}\,\bar{n} \\ \bar{a}\,\bar{b}\,c \\ a\,\bar{b}\,\bar{c} \end{matrix}\right\} \qquad \left.\begin{matrix} \bar{l}\,\bar{m}\,\bar{n} \\ \bar{a}\,b\,\bar{c} \\ a\,\bar{b}\,\bar{c} \end{matrix}\right\}$$

which give the following values:—

$$\begin{array}{lll} x_0 = -a, & y_0 = -b, & z_0 = -c, \\ x_1 = -a, & y_1 = \dfrac{bmn\,(a-l)}{l\,(bn+cm)}, & z_1 = \dfrac{cmn\,(a-l)}{l\,(bn+cm)}, \\ x_2 = \dfrac{aln\,(b-m)}{m\,(an+cl)}, & y_2 = -b, & z_2 = \dfrac{cln\,(b-m)}{m\,(an+cl)}, \\ x_3 = \dfrac{alm\,(c-n)}{n\,(am+bl)}, & y_3 = \dfrac{blm\,(c-n)}{n\,(am+bl)}, & z_3 = -c. \end{array}$$

The other secondary pyramids are obtained from the first by changing simultaneously the signs of any two of $a\,b\,c$ and of the corresponding pair of $l\,m\,n$. Now, changing the signs of $a\,b\,l\,m$ and leaving the signs of $c\,n$ unchanged is equivalent to changing the signs of x and y and retaining those of z and of all the constants, and so with regard to the other two permutations. Hence it is clear that the sum of the sixteen x's is identically equal to zero, and so of the y's and z's. It follows, then, that if these sixteen points be considered as the centres of spheres of equal weight, the centre of gravity is at the origin. If, therefore, the four secondary pyramids have all the same volume, the proposition will be proved.

Consider the hexahedron (or parallelepiped)

$$a\,0\,0,\quad 0\,b\,0,\quad 0\,0\,c,$$
$$\bar{a}\,0\,0,\quad 0\,\bar{b}\,0,\quad 0\,0\,\bar{c},$$

the twelve diagonals of the faces of which are the edges of the two tetrahedra $(a\,b\,c)$ and $(\bar{a}\,\bar{b}\,\bar{c})$. Let us suppose that its alternate corners are cut off by the four planes $\bar{l}\,\bar{m}\,\bar{n}$, $\bar{l}\,m\,n$, $l\,\bar{m}\,n$, $l\,m\,\bar{n}$.

These planes will cut off equal lengths upon the corresponding edges of the hexahedron, and therefore the pyramids cut off will be of equal volume, as may easily be shown. Next, if we consider each plane of the type $(l\,m\,n)$ as the common base of two pyramids, one cut off from the hexahedron, and the other from the original tetrahedron $(a\,b\,c)$, we see that these have the same altitude, and that the proportion between their bases is the same at all the four corners. Hence the pyramids cut off from the tetrahedron are also all equal *inter se*. But this being the case, it has already been shown that their centre of figure is at the origin. Hence the centre of figure of the solid about the origin, that is, of the remainder of the tetrahedron $(a\,b\,c)$, is also at the origin.

I have chosen the simplest case for demonstration. The general proposition may be inferred from the principle of the permanence of equivalent forms, or, with a little more trouble in picking out the details, may be proved independently.

COR. 1.—The solid formed about the origin by any number of such intersecting tetrahedra, and by hexahedra of the same system, has also the origin for its centre of figure.

COR. 2.—Cleavage parallel to one of the coordinate planes simply shifts the centre of figure along the corresponding coordinate axis.

COR. 3.—If the solid be made up of any number of tetrahedra and hexahedra about the same axes and having their corners on the same four dia-

gonals, then the effect of cleavage in a tetrahedral plane is simply to shift the centre of figure along the diagonal corresponding to the cleavage plane. If there are to be no hollow spaces, there must of course be at most only one hexahedron, and one direct and one inverse tetrahedron.

Octohedral cleavage is thus seen to be more complex than hexahedral, as might be expected.

NOTE ON QUESTION 2740: BY PROFESSOR CAYLEY.

The envelope of the curve

$$A\ \cos 2\theta + B \sin 2\theta + C \cos \theta + D \sin \theta + E = 0,$$

(where A, B, C, D, E are any functions of the coordinates, and θ is a variable parameter,) is obtained in the particular case $E=0$ (Salmon's *Higher Plane Curves*, p. 116), and the same process is applicable in the general case where E is not $=0$. From the great variety of the problems which depend upon the determination of such an envelope, the result is an important one, *and ought to be familiarly known* to students of analytical geometry. We have only to write $z = \cos\theta + i \sin\theta$, the trigonometrical functions are then given as rational functions of z, and the equation is converted into a quartic equation in z; the result is therefore obtained by equating to zero the discriminant of a quartic function. The equation, in fact, becomes

$$A\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)+B\frac{1}{2i}\left(z^2-\frac{1}{z^2}\right)+C\frac{1}{2}\left(z+\frac{1}{z}\right)+D\frac{1}{2i}\left(z-\frac{1}{z}\right)+E=0,$$

that is, $\quad A(z^4+1)-Bi(z^4-1)+C(z^3+z)-Di(z^3-z)+2Ez^2=0;$

or, multiplying by 12 to avoid fractions, this is

$$(a, b, c, d, e)(z, 1)^4 = 0,$$

where $\quad a = 12(A-Bi), \quad b = 3(C-Di), \quad c = 4E,$

$$e = 12(A+Bi), \quad d = 3(C+Di);$$

and substituting in

$$(ae-4bd+3c^2)^3-27(ace-ad^2-b^2e+2bcd-c^3)^2=0,$$

the equation divides by 1728, and the final result is

$$\left\{12(A^2+B^2)-3(C^2+D^2)+4E^2\right\}^3$$
$$-\left\{27A(C^2-D^2)+54BCD-[72(A^2+B^2)+9(C^2+D^2)]E+8E^3\right\}^2=0.$$

It is to be noticed, that in developing the equation according to the powers of E, the terms in E^6, E^4 each disappear, so that the highest power is E^3; the degree in the coordinates, or order of the curve, is on this account sometimes lower than it would otherwise have been.

2823 (Proposed by Professor SYLVESTER.)—Show that on a chess-board the chance of a rook moving from one square to another without changing

colour is $\frac{3}{7}$; but that without altering the equality of the *number* of the black and white squares, but only the manner of their distribution, the chance may be made equal to $\frac{1}{2}$.

I. *Solution by* MORGAN JENKINS, M.A.

Wherever the rook may be placed on the board, there are in each of the two lateral directions in which the piece can move 3 squares of the same and 4 of a different colour to that of the square on which the rook stands. The required chance is therefore $\frac{3}{7}$.

No possible distribution of the colours can be made which shall raise the chance to $\frac{1}{2}$ *wherever the rook may be placed;* for if x be the number of black squares in a row, then out of the 14 squares to which the rook may move 7 squares must be of the same and 7 of different colour to that of the square occupied by the rook. Hence, placing the rook on any of the x black squares, we should have $x-1$ other black squares in the same row, and therefore $8-x$ other black squares in the same column; and placing the rook on any of the $8-x$ white squares, we should have x squares in the same row, and therefore $7-x$ black squares in the same column, whence follows the equation

$$x(9-x)+(8-x)(7-x) = 32 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The only solution of this is $x=4$; which shows that there could be only 4 black squares in any row, and in like manner in any column; but this is inconsistent with the supposition in (1), that if there are x or 4 black squares in a row, there will be $9-x$ or 5 in some of the columns.

Hence the chance cannot be $\frac{1}{2}$ for every position of the rook; but by supposing that every position of the rook is equally probable, we may, by various distributions of the colours, make the total chance for all positions of the rook $\frac{1}{2}$.

Let $x_1, x_2, \ldots\ldots x_8$ be the number of black squares in the rows,

$y_1, y_2, \ldots\ldots y_8$ „ „ „ columns;

then $$x_1+x_2+\ldots\ldots+x_8 = 32, \quad y_1+y_2+\ldots\ldots+y_8 = 32 \quad\ldots\ldots (2, 3);$$

and also the following equation is easily obtainable,

$$\left.\begin{array}{l} x_1(x_1-1)+x_2(x_2-1)+\ldots\ldots+(8-x_1)(7-x_1)+\ldots\ldots \\ +y_1(y_1-1)+y_2(y_2-1)+\ldots\ldots+(8-y_1)(7-y_1)+\ldots\ldots \end{array}\right\} = 32\times 14,$$

which reduces to $$x_1^2+\ldots\ldots+y_1^2+\ldots\ldots = 16\times 18 \quad\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

It is easily seen that, if $x_1, x_2, \ldots\ldots, y_1, y_2, \ldots\ldots$ be a set of solutions, then $8-x_1, 8-x_2, \ldots\ldots, 8-y_1, 8-y_2, \ldots\ldots$ is another set, which may be called a *complementary set.*

To facilitate the finding of solutions of (2), (3), and (4), I have *made the assumption* (involving possibly a loss of generality) that

$$x_1^2+x_2^2+\ldots\ldots+x_8^2 = 16\times 9 = y_1^2+y_2^2+\ldots\ldots+y_8^2 \quad\ldots\ldots (5, 6).$$

Supposing that there are a rows containing 1 black square, b rows containing 2 black squares, and so on, we have

$$a+b+c+\ldots = 8, \quad a+2b+3c+\ldots = 32, \quad a+4b+9c+\ldots = 144 \quad\ldots\ldots (7, 8, 9);$$

whence we obtain

$$b+3c+6d+10e+\ldots = 56, \quad c+3d+6e+\ldots = 32 \quad\ldots\ldots (10, 11).$$

The following sets of values of $x_1, x_2, \ldots . x_8$ will be found, I think, to be all which can be obtained from the positive integral solutions of (7), (10), and (11).

[6, 6, 4, 4, 4, 4, 2, 2]	self-complementary
[6, 6, 5, 4, 3, 3, 3, 2]	[6, 5, 5, 5, 4, 3, 2, 2]
[7, 5, 4, 4, 4, 3, 3, 2]	[6, 5, 5, 4, 4, 4, 3, 1]
[7, 5, 5, 3, 3, 3, 3, 3]	[5, 5, 5, 5, 5, 3, 3, 1].

The following squares are specimens of distributions obtainable from these sets of values :—

4	W	W	W	W	B	B	B	B
4	W	W	W	W	B	B	B	B
2	W	W	W	W	B	B	W	W
2	W	W	W	W	B	B	W	W
4	B	B	B	B	W	W	W	W
4	B	B	B	B	W	W	W	W
6	B	B	B	B	B	B	W	W
6	B	B	B	B	B	B	W	W
	4	4	4	4	6	6	2	2

2	W	W	W	W	W	B	W	B
4	B	W	W	W	W	B	B	B
3	B	W	W	W	W	W	B	B
3	B	B	B	W	W	W	W	W
3	B	B	B	W	W	W	W	W
5	B	B	B	B	B	W	W	W
6	B	B	B	B	B	W	B	W
6	B	B	B	B	B	B	W	W
	7	5	5	3	3	3	3	3

[Professor SYLVESTER remarks that any anallagmatic distribution of the black and white squares will satisfy the condition. For, in such a square, whether the motion is from a row to a row or from a column to a column, there will by its definition be as many passages from like colour to like as from a colour to its opposite. Suppose an anallagmatic square is made isochromatic, *i. e.*, having always the same difference (positive or negative) between the black and white colours in each row and column (see the diagram at the end of Vol. X. of the *Reprint*). Then, putting n for the number of squares in a side, and x, $n-x$ for the number of the separate colours respectively, it is clear from the above solution that we must have

$$\Sigma x(n-x) = \tfrac{1}{2}\Sigma n . \frac{n-1}{2}, \quad \therefore\ x(n-x) = \frac{n^2-n}{4}, \quad \text{i.e.,}\ x = \frac{n \pm \sqrt{n}}{2},$$

which proves that an anallagmatic square cannot be made isochromatic unless the number of places it contains is a perfect fourth power. This disposition springs directly from an improved form given to NEWTON'S complete rule for limiting the total number of imaginary roots in an algebraic equation.]

II. *Solution by* MATTHEW COLLINS, B.A.

1. As each line and column of a chessboard contains 4 white and 4 black squares, therefore upon whatever square a rook be placed there are 3 other squares of the same colour, and 7 other squares altogether in the line or column in which the rook can move, and hence his chance of moving into a square of the same colour as that he left is plainly $= \frac{3}{7}$.

2. If the line bisecting two opposite sides of the chessboard be the boundary between the white and the black squares, it is plain the square common to any line and column has, upon that line and column taken jointly, 10 other squares of the same colour as itself; and as there are 14 other squares altogether upon that line and column, hence in this case the chance of a rook moving into a square of the same colour as that it left is $= \frac{10}{14} = \frac{5}{7}$.

3. If the diagonal divides the white from the black squares, and if the squares through which it passes be white and black alternately, the required chance of a rook not changing colour when moved would plainly be

$$\frac{14.7+10.9+6.11+2.13}{32.14}=\frac{5}{8}.$$

4. But if the four white squares on the diagonal occupy half of it, the other half running through four black squares, then the required chance is

$$\frac{1}{32.14}(4.6+6.7+6.8+5.9+5.10+3.11+2.12+1.13)=\frac{279}{448}.$$

5. Lastly, if one row consist of eight white squares, and another row of eight black squares, the remaining six rows of the chess-board being left unchanged, the required chance is then plainly

$$\frac{8.10+24.6}{32.14}=\frac{1}{2}.$$

2741. (Proposed by S. TEBAY, B.A.)—A moveable event (depending on the moon) happened in the year y; show that it will happen again in the year $y+19(28t-5a+8b)$; where a may be 0, 1, 2, 3; b the number of completed centuries since the year y which are not leap years; and t any arbitrary integer.

Solution by the PROPOSER.

Let $Y=y+4n-a$ be the year; then the number of days in this interval is $\quad 365(4n+a)+n-b=7(208n+52a)+5n+a-b$;

therefore $5n+a-b$ must be divisible by 7;

therefore $\quad n=7m-3a+3b,\quad$ and $\quad 4n+a=28m-11a+12b,$

which must be divisible by 19;

therefore $\quad m=19t-3a+5b,\quad \therefore\quad 4n+a=19(28t-5a+8b),$

and $\quad Y=y+19(28t-5a+8b).$

Example: Easter-day cannot happen later than April 25th; the last time this took place was in the year 1734; when will it happen again? Let $t=0$, $a=0$, $b=1$; therefore $Y=1886$, the year required.

2598. (Proposed by G. M. SMITH, B.A.)—A uniform rod (mass $=\mu$) is placed inside a spherical shell (mass $=m$, radius $=a$) which, under the influence of gravity, rolls down the surface of a rough sphere (radius $=r$); find equations for the movements of the rod and shell, there being no friction

between them; and hence show that, if a particle μ is placed near the lowest point of a spherical shell (m) which performs small oscillations on a rough horizontal plane, the length of the simple equivalent pendulum is $\frac{2m}{2m+\mu}\,a$.

Solution by the PROPOSER.

Let the horizontal and vertical lines through the centre O of the sphere be taken as axes of x and y respectively; let (x, y) be the coordinates of the centre C of the shell; ψ = angle through which the shell has turned; ϕ = angle between radius and Oy; θ = COy; c = perpendicular from C on rod. Then

$$m\left(\frac{d^2x}{dt^2}\,\delta x+\frac{d^2y}{dt^2}\,\delta y+a^2\frac{d^2\psi}{dt^2}\,\delta\psi\right)+\mu\,.\left\{\left(\frac{d^2x}{dt^2}-c\,\frac{d^2\cos\phi}{dt^2}\right)\left(\delta x-c\delta\cos\phi\right)\right.$$
$$\left.+\left(\frac{d^2y}{dt^2}-c\,\frac{d^2\sin\phi}{dt^2}\right)\left(\delta y-c\delta\sin\phi\right)+k^2\frac{d^2\phi}{dt^2}\,\delta\phi\right\}$$
$$=-(m+\mu)\,g\,\delta y+\mu cg\cos\phi\,\delta\phi.$$

Substituting for x and y their values in terms of θ, and selecting the coefficients of $\delta\theta$ and $\delta\phi$, we obtain, by equating these coefficients separately to zero, the equations

$$(2m+\mu)(a+r)\frac{d^2\theta}{dt^2}-\mu c\left\{\cos(\phi-\theta)\frac{d^2\phi}{dt^2}-\sin(\phi-\theta)\left(\frac{d\phi}{dt}\right)^2\right\}$$
$$+(m+\mu)\,g\cos\theta=0,$$

$$c(a+r)\left\{\cos(\phi-\theta)\frac{d^2\theta}{dt^2}+\sin(\phi-\theta)\left(\frac{d\theta}{dt}\right)^2\right\}-(k^2+c^2)\frac{d^2\phi}{dt^2}+gc\cos\phi=0.$$

In the particular case in which the radius becomes a particle, and the sphere a horizontal plane, these equations become

$$(2m+\mu)\,a\,\frac{d^2\psi}{dt^2}+\mu c\,\frac{d^2\cos\phi}{dt^2}=0,$$

$$ac\sin\phi\,\frac{d^2\psi}{dt^2}-(k^2+c^2)\frac{d^2\phi}{dt^2}+cg\sin\phi=0,$$

which give (by putting $\phi=\frac{1}{2}\pi-\beta$, where β is small)

$$\frac{\mu a^2}{\mu+2m}\cdot\frac{d^2\beta}{dt^2}-a^2\frac{d^2\beta}{dt^2}=ga\beta,\quad\text{or}\quad\frac{2m}{2m+\mu}\cdot\frac{d^2\beta}{dt^2}=g\beta,$$

showing that the length of the simple pendulum $=\frac{2m}{2m+\mu}\,a$.

In the case in which the sphere reduces to a plane of inclination $= i$, our equations become

$$(2m+\mu)\,a\,\frac{d^2\psi}{dt^2}+\mu c\,\frac{d^2\cos(\phi+i)}{dt^2}+(m+\mu)\,g\sin i=0,$$

$$ac\sin(\phi+i)\frac{d^2\psi}{dt^2}-(k^2+c^2)\frac{d^2\phi}{dt^2}+gc\cos\phi=0.$$

2221. (Proposed by W. S. BURNSIDE, M.A.)—1. Show how to determine the locus of the feet of perpendiculars from a fixed point on the generating lines of a system of confocal quadrics.

2. Prove that this locus is described by foci of the plane sections passing through the fixed point.

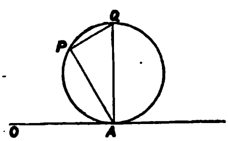
Solution by the PROPOSER.

1. Since the foot of a perpendicular on a generator is the foot of a perpendicular on a tangent plane through that generator, the question (1) is reduced to first finding the locus of feet of perpendiculars on tangent planes to one surface of the system, and then eliminating the arbitrary parameter which enters by means of the equation of the surface itself.

2. Consider the section of a surface of the system by a plane P, and let one of the foci of the section be at F; this being so, reciprocating origin at F, the section of the reciprocal surface by the plane P is a circle; consequently the section of the asymptotic cone of this surface is a circle; but the asymptotic cone is similar and similarly placed to the cone generated by lines passing through F perpendicular to the tangent planes through F to the given surface; therefore one of the focal lines of this cone is the perpendicular to the plane P at F. Finally the focal lines of the tangent cone are the generators of the confocal hyperboloid of one sheet through the vertex.

2795. (Proposed by S. TEBAY, B.A.)—Find the average square of the distance between the centres of the inscribed and circumscribed circles of a triangle inscribed in a given circle.

I. *Solution by the* Rev. J. WOLSTENHOLME, M.A.



If a be the radius of the given circle, x radius of an inscribed circle, square of distance between the centres is a^2-2ax, and the question is equivalent to finding the average value of x. Now, considering one point A of the triangle fixed, which is clearly allowable; AO the tangent at A; P, Q the other angular points of the triangle,

$$\angle \text{OAP} = \theta, \quad \angle \text{OAQ} = \phi, \quad (\phi > \theta), \quad x = 4a \sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}(\phi-\theta) \cos \tfrac{1}{2}\phi,$$

$$\therefore \text{ average value of } x = a \frac{\int_0^\pi \int_\theta^\pi 4 \sin \frac{1}{2}\theta \sin \frac{1}{2}(\phi-\theta) \cos \frac{1}{2}\phi \, d\theta \, d\phi}{\int_0^\pi \int_\theta^\pi d\theta \, d\phi}$$

$$= a \frac{6-\frac{1}{2}\pi^2}{\frac{1}{2}\pi^2} = a\left(\frac{12}{\pi^2}-1\right)$$

(See *Book of Mathematical Problems*, Ex. 1036);

therefore average square of distance between the centres is $3a^2\left(1-\dfrac{8}{\pi^2}\right)$.

II. *Solution by the* PROPOSER.

Let D be the distance between their centres; then

$$D^2 = R^2 - 2Rr$$
$$= R^2(1 - 8\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C)$$
$$= R^2 - 4R^2\sin\tfrac{1}{2}A\left\{\sin\tfrac{1}{2}(A+2B) - \sin\tfrac{1}{2}A\right\}.$$

Supposing the vertical angle to remain constant while B varies from 0 to $\pi - A$, we find the sum of the squares of the distances

$$= 2\int_0^{\pi}\int_0^{\pi-A} D^2\,dA\,dB = 3R^2(\pi^2-8).$$

The number of triangles $= 2\int_0^{\pi}\int_0^{\pi-A} dA\,dB = \pi^2.$

Hence the average square of the distance $= 3R^2\left(1 - \dfrac{8}{\pi^2}\right).$

2797. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—In any complete quadrilateral, the radical axis of the three circles whose diameters are the diagonals will pass through the centres of perpendiculars of the four triangles formed by the four straight lines.

Solution by MATTHEW COLLINS, B.A.

Let ABCD (Fig. 1) be the quadrilateral; then the circles whose diameters are the three diagonals AC, BD, EF will have their centres at the middle points G, H, K of these diagonals; the three altitudes of the triangle ABF, viz., AOA′, BOB′, FOF′, meet in one point O (its ortho-centre); therefore Gg, perpendicular to AOA′, is parallel to BC and bisects AA′ in g; similarly, Hh perpendicular to BB′, and Kk perpendicular to FF′, bisect BB′ and FF′ in h and k.

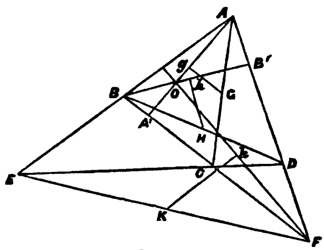


FIG. 1.

Now, the square of the tangent drawn from O to the circle (G) $= OG^2 - AG^2 = Og^2 - Ag^2 = OA \cdot OA'$; similarly, the square of the tangent from O to the circle (H) $= OH^2 - BH^2 = Oh^2 - Bh^2 = OB \cdot OB'$, and the square of the tangent from O to the circle (K) $= OK^2 - FK^2 = Ok^2 - Fk^2 = OF \cdot OF'$.

But $OA \cdot OA' = OB \cdot OB' = OF \cdot OF'$, therefore the tangents from O to the three circles are equal to each other, consequently the ortho-centre O of the triangle ABF is a point on the radical axis of these three circles. For like reasons, the ortho-centres of the other three triangles formed by three

other sides of the quadrilateral ABCD (viz., the triangles ADE, BCE, CDF) lie also upon the same radical axis.

COROLLARY.—As the three circles (G), (H), (K) are proved coaxal, their centres, viz., the middle points of the three diagonals of the complete quadrilateral ABCDEF, lie in one straight line. A simple proof of this theorem may also be obtained as follows:—

Let ABCDEF (Fig. 2) be the complete quadrilateral; AC, BD, EF its three diagonals. Through B, C, D, E, F draw lines parallel to the sides of the angle A, forming the parallelograms ABGD, AEHF, &c.

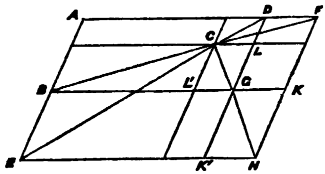


FIG. 2.

Then (by Euclid I. 43), parallelogram CK = AC = CK′, and therefore parallelogram LK = L′K′; hence (by Euclid I. 43, *ex absurdo*) the points C, G, H lie in one straight line.

Now, as the diagonals of a parallelogram bisect each other, the middle point of the diagonal BD is the same as the middle point of AG; and, for a like reason, the middle point of the third diagonal EF is the same as the middle point of AH. But the middle points of AC, AG, AH lie plainly in one straight line parallel to CGH; therefore the middle points of the three diagonals AC, BD, EF of the complete quadrilateral ABCDEF lie in one straight line.

NOTE.—Several new and curious theorems can be easily obtained from the foregoing theorem and its corollary by means of the methods of *reciprocation* and *inversion*.

2622. (Proposed by A. W. PANTON, B.A.)—The equation connecting the distances (r_1, r_2, r_3) of any point on a Cartesian oval from the foci is

$$(\beta-\gamma)\,\alpha^{\frac{1}{2}} r_1 + (\gamma-\alpha)\,\beta^{\frac{1}{2}} r_2 + (\alpha-\beta)\,\gamma^{\frac{1}{2}} r_3 = 0,$$

where α, β, γ are the distances of the foci from the triple focus.

Solution by W. S. MCCAY, B.A.

The Cartesian oval may be written

$$lr_1 + mr_2 + nr_3 = 0, \text{ or } \lambda r_1 + \mu r_2 = d \;\ldots\ldots\ldots\ldots (1, 2).$$

Now to reduce the first of these to the second by means of the identity

$$Ar_1^2 + Br_2^2 + Cr_3^2 = D \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

(where A, B, C are the distances between the foci, and D = ABC) we get by substituting r_3 in (3) from (1)

$$Ar_1^2 + Br_2^2 + C\left(\frac{lr_1 + mr_2}{n}\right)^2 = D.$$

That this may be identical with (2), we must have

$$\frac{l^2}{\mathrm{A}} + \frac{m^2}{\mathrm{B}} + \frac{n^2}{\mathrm{C}} = 0, \quad \text{also} \quad \lambda^2 : \mu^2 = \frac{\mathrm{A}n^2 + \mathrm{C}l^2}{\mathrm{B}n^2 + \mathrm{C}m^2} \;\ldots\ldots\ldots\ldots (5, 6).$$

The last equation becomes by means of (5) $\lambda^2 : \mu^2 = \dfrac{\mathrm{A}^2}{l^2} : \dfrac{\mathrm{B}^2}{m^2}$ .... (7).

Now since (2) may be written

$$\left(\lambda^2 r_1^2 - \mu^2 r_2^2 - \frac{\lambda^2 + \mu^2}{\lambda^2 - \mu^2}\right)^2 + \frac{4d^2 \lambda^2 \mu^2}{\lambda^2 - \mu^2}\left(r_1^2 - r_2^2 + \frac{d^2}{\lambda^2 - \mu^2}\right) = 0,$$

it follows that if (α, β, γ) are the distances of the foci from the triple focus $\left(\text{centre of circle } \lambda^2 r_1^2 - \mu^2 r_2^2 = \dfrac{\lambda^2 + \mu^2}{\lambda^2 - u^2}\right)$, $\lambda^2 \alpha - \mu^2 \beta = 0$.

Hence (7) becomes $\quad l : m = \mathrm{A}\alpha^{\frac{1}{2}} : \mathrm{B}\beta^{\frac{1}{2}}$;

so too $\quad m : n = \mathrm{B}\beta^{\frac{1}{2}} : \mathrm{C}\gamma^{\frac{1}{2}}$;

therefore $\quad l : m : n = (\beta - \gamma)\,\alpha \;\; : (\gamma - \alpha)\,\beta^{\frac{1}{2}} : (\alpha - \beta)\,\gamma^{\frac{1}{2}}$.

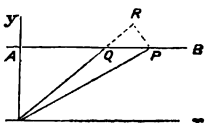
2670. (Proposed by C. T. HUDSON, LL.D.)—An observer, seated in the aisle of a cathedral and looking westward, sees the horizontal lines above the arches converging to a point before him, and consequently cutting the horizon at a certain angle. On his facing northwards, he sees that the same lines are now parallel to the horizon. Required the curve that they will appear to lie in, as he gradually turns his head through 90°.

Solution by the PROPOSER.

This problem is indeterminate; for the plane on which the given lines are seen projected at any instant, will at that instant be referred to various distances according to the fancy of each observer.

(i.) Let this plane be supposed to be vertical, and always at a fixed distance from the observer; then the ultimate intersections of its various positions will be a vertical cylinder of which the axis passes through the observer, and a plane through his eye and the given horizontal lines will cut the cylinder in an ellipse, which therefore is the required curve.

(ii.) Let that portion of the given horizontal line which is seen distinctly at any instant be supposed to be referred to its real distance from the observer at that instant.



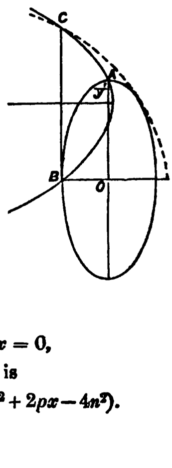
Let O be the observer, AB the given horizontal line, Oy and Ox lines perpendicular and parallel to AB in the plane drawn through AB and O, QP the portion of AB which is seen distinctly when the observer is looking along OP, but which is seen projected into the position PR at right angles to OP. Let also AO $= c$, PO$x = \theta$; then the equation to PR is $\quad x \cos\theta + y \sin\theta = c \operatorname{cosec}\theta$;

and the ultimate intersections of all its positions obtained by varying θ lie in the curve $$x^2 = 4c(c-y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The sensation of the curvature of AB is, according to this hypothesis, produced by the attention being fixed on the extremity of the line RP, which always moves so as to touch the parabola (1), and at the same time remain at right angles to the visual ray.

2762. (Proposed by the EDITOR.)—Find the envelope of an ellipse which has one vertex in the curve of a given parabola, and touches at its adjacent vertex the extremity of a fixed double-ordinate of the parabola.

I. *Solution by* J. J. WALKER, M.A.



Taking the ordinate of the parabola and the axis of the ellipse which passes through the vertex of contact as axes of y and x, the equation to the parabola will be (n being the fixed ordinate)

$$y^2 - 2ny + px = 0 \ldots\ldots\ldots\ldots (1),$$

and that of the ellipse

$$\frac{y^2}{(y'+n)^2} + \frac{p^2x^2}{(y'^2-n^2)^2} + \frac{2px}{y'^2-n^2} = 0 \ldots. (2),$$

where y' is the ordinate of the parabola passing through the first vertex of the ellipse, since the axes of the ellipse will plainly be $y'+n$ and $\dfrac{n^2-y'^2}{p}$. Equation (2), when multiplied by $(y'^2-n^2)^2$ and arranged by powers of y', becomes

$$(y^2+2px)\,y'^2 - 2ny^2y' + n^2y^2 + p^2x^2 - 2pn^2x = 0,$$

the discriminant of which, with respect to y' as variable, is

$$(y^2+2px)(n^2y^2+p^2x^2-2pn^2x) - n^2y^4, \quad \text{or} \quad p^2x^2(y^2+2px-4n^2).$$

Hence the required envelope is the parabola

$$y^2 + 2px - 4n^2 = 0,$$

which has as ordinate the fixed double ordinate, and a parameter double that of the given parabola.

II. *Solution by the* REV. J. WOLSTENHOLME, M.A.; S. BILLS; *and others.*

The equation of the parabola being $y^2 = 4ax$, let the coordinates of the fixed vertex be $a\alpha^2$, $2a\alpha$, and of the moving vertex $a\lambda^2$, $2a\lambda$; then the equation of the ellipse is $$\frac{(x-a\lambda^2)^2}{a^2(\alpha^2-\lambda^2)^2} + \frac{(y-2a\alpha)^2}{4a^2(\alpha-\lambda)^2} = 1,$$

or $$4(x-a\lambda^2)^2 + (\alpha+\lambda)^2(y-2a\alpha)^2 = 4a^2(\alpha^2-\lambda^2)^2,$$

or $$4(x-a\alpha^2)(x+a\alpha^2-2a\lambda^2)+(\alpha+\lambda)^2(y-2a\alpha)^2=0;$$
and the envelope is
$$\alpha^2(y-2a\alpha)^4=\left\{(y-2a\alpha)^2-8a(x-a\alpha^2)\right\}\left\{\alpha^2(y-2a\alpha)^2+4(x^2-a^2\alpha^4)\right\},$$
or $$(x-a\alpha^2)^2\left\{(y-2a\alpha)^2-8a(x+a\alpha^2)\right\}=0.$$

The required envelope is thus seen to consist of the fixed double ordinate, and a parabola similarly situated to the given one, of twice the dimensions, and having its vertex in the diameter through the given vertex. For the contact to be real, the vertices must be on opposite sides of the axis of the parabola.

2800. (Proposed by J. Wilson.)—Three circles passing through a point P form a circular triangle ABC, and each side of this circular triangle or its continuation is cut orthogonally at the point P′ by a circle passing through P; prove that the three circles described about the triangles PAP′, PBP″, PCP‴ are co-axal.

Solution by Archer Stanley.

By inverting the figure with respect to the point P, it will readily be found that the theorem is the inverse of the well-known one, that the perpendiculars are concurrent which are drawn from the vertices of a triangle to the respectively opposite sides.

2688. (Proposed by J. Griffiths, M.A.)—A variable triangle circumscribes an equilateral hyperbola, and is such that its *nine-point* circle passes through the centre of this curve; prove that the locus of the centre of its *circumscribing* circle is the hyperbola in question.

Solution by the Rev. J. Wolstenholme, M.A.

If an equilateral hyperbola be inscribed in a triangle, the centre of the hyperbola lies on the polar circle of the triangle: if, then, the centre lies on the nine-point circle, it must lie on the circumscribed circle, since the three circles are coaxal. Let O be this centre, ABC the triangle, and let the asymptotes of the hyperbola meet the circumscribed circle in P, Q. Then, since OPQ, ABC are triangles in the same conic (circle), their sides will touch one conic; but five of them touch the hyperbola, therefore also the sixth (PQ) touches the hyperbola: it will be bisected in the point of contact, which will therefore be the centre of the circumscribed circle of the triangle. Hence, if a rectangular hyperbola touch the sides of a triangle, and its centre lie on the nine-point circle of the triangle, the centre of the circumscribed circle lies on the hyperbola; which proves the theorem.

2654. (Proposed by W. S. BURNSIDE, M.A.)—Determine the polar reciprocal of the quartic curve

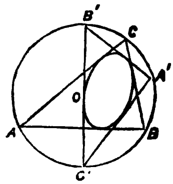
$$y^2z^2+z^2x^2+x^2y^2+2xyz\,(lx+my+nz)=0.$$

Solution by the PROPOSER.

Write $\dfrac{1}{X}, \dfrac{1}{Y}, \dfrac{1}{Z}$ for x, y, z, and the quartic is transformed to $X^2+Y^2+Z^2+2lYZ+2mZX+2nXY=0$, and the polar of a point $\alpha\beta\gamma$ becomes $\alpha YZ+\beta ZX+\gamma XY=0$. So the question is reduced to finding when these two conics touch, which is easily done.

2726. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—If a conic touch the sides of a triangle and pass through the centre of the circumscribed circle, the director circle of the conic will touch the circumscribed circle of the triangle.

Solution by W. S. MCCAY, B.A.



If the conic be $(lx)^{\frac12}+(my)^{\frac12}+(nz)^{\frac12}=0$, it follows from the general equation of the director circle given at p. 338 of Salmon's *Conics*, that the radical axis of director circle and circumscribing circle of the triangle is $\quad l\cot A\,.\,x+m\cot B\,.\,y+n\cot C\,.\,z=0;$ and if this touch the circumscribing circle $(ayz+bxz+cxy=0)$, we must have

$$l^2\cos^2 A+m^2\cos^2 B+n^2\cos^2 C-2m^2n^2\cos B\cos C$$
$$-2l^2n^2\cos B-2m^2l^2\cos^2 C=0,$$

or $\qquad (l\cos A)^{\frac12}+(m\cos B)^{\frac12}+(n\cos C)^{\frac12}=0.$

But this is the condition that centre of circumscribing circle ($\cos A$, $\cos B$, $\cos C$) should lie on the conic.

[Mr. WOLSTENHOLME'S proof is as follows:—
Let O be the centre of the circumscribed circle, and let a tangent to the conic at O meet the circle in B′, C′; then the tangents from B′, C′ to the conic will meet in a point (A′) which will lie on the circle, since ABC, A′B′C′ are triangles circumscribing the same conic. These tangents being at right angles, A′ is a point on the director circle; and since B′C′ is bisected in O, OA′ passes through the centre of the conic, that is, through the centre of the director circle. The circles therefore touch at A′.]

2770. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—If a conic be drawn through S, S′, the foci (both real or both impossible) of a given conic, and osculate the conic at a point P, the tangents at S, S′ will intersect in the centre of curvature at P.

Solution by the PROPOSER.

The equation of any conic touching the given conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(a\cos\theta,\ b\sin\theta)$ is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\left(\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}-1\right)\left(\mathrm{L}\cos\theta\,\frac{x}{a}+\mathrm{M}\sin\theta\,\frac{y}{b}-\mathrm{N}\right)=0;$$

if this osculate at the point θ, $\quad \mathrm{L}\cos^2\theta+\mathrm{M}\sin^2\theta=\mathrm{N}$;

if it also pass through the two foci on the axis of x,

$$\mathrm{L}+\mathrm{N}=0,\quad \left(1-\frac{b^2}{a^2}\right)(1+\mathrm{L}\cos^2\theta)=1-\mathrm{N};$$

Also the pole of SS′ is given by the equations

$$2(1+\mathrm{L}\cos^2\theta)\frac{x}{a}+\frac{\sin\theta\cos\theta}{b}(\mathrm{L}+\mathrm{M})y=0,\quad \frac{\mathrm{M}+\mathrm{N}}{b}y\sin\theta+2(1-\mathrm{N})=0.$$

But, from the former equations,

$$-\mathrm{L}=\mathrm{N}=\frac{b^2}{a^2\sin^2\theta+b^2\cos^2\theta}=\frac{\mathrm{M}\sin^2\theta}{1+\cos^2\theta};$$

therefore $\frac{\mathrm{M}+\mathrm{N}}{1-\mathrm{N}}=\frac{2b^2}{(a^2-b^2)\sin^4\theta}$, therefore $y=-\frac{a^2-b^2}{b}\sin^3\theta$;

and $$\frac{\mathrm{L}+\mathrm{M}}{1+\mathrm{L}\cos^2\theta}=\frac{2b^2\cos^2\theta}{a^2\sin^4\theta},$$

therefore $$\frac{x}{a}=\frac{b\cos^3\theta}{a^2\sin^3\theta}\cdot\frac{a^2-b^2}{b}\sin^3\theta=\frac{a^2-b^2}{a^2}\cos^3\theta;$$

or the tangents at S, S′ intersect in the point $\left(\frac{a^2-b^2}{a}\cos^3\theta,\ \frac{b^2-a^2}{b}\sin^3\theta\right)$, the centre of curvature at P.

It may be noticed that, in any conic passing through SS′ and touching at P, the tangent and normal at P, form with PS, PS′ a harmonic pencil; therefore SS′ and the normal at P are conjugates with repect to this conic, or the tangents at S, S′ intersect on the normal at P.

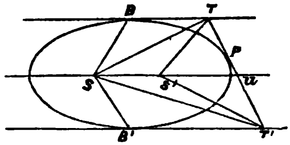
2799. (Proposed by A. MARTIN.)—If n dice are thrown, what is the chance of an odd number of aces turning up?

Solution by the REV. J. WOLSTENHOLME, M.A.

The chance is obviously $\frac{6^n-4^n}{2\times 6^n}$. If the dice be regular tetrahedra, the chance is $\frac{1}{2}-\frac{1}{2^{n+1}}$.

2789. (Proposed by W. S. M'CAY, B.A.)—If two circles pass through the foci of an ellipse and touch the same variable tangent to the curve, the angle at which they intersect is constant and equal to $2\tan^{-1}\dfrac{b}{c}$.

Solution by the Rev. J. WOLSTENHOLME, M.A.



Let SS′ be foci of an ellipse, TPT′ a tangent meeting the tangents at BB′, the ends of the minor axis in TT′, and the major axis in U. Then $\angle$UTS′ = $\angle$BTS = $\angle$TSS′; therefore triangles UTS′, UST are similar, and UT : US′ = US : UT, or US . US′ = UT²; therefore the circle through STS′ will touch the tangent at T, so the circle through ST′S′ will touch the tangent at T′; and the angle at which these circles cut in S, S′ is the sum of the angles STS′, ST′S′, and therefore the difference of the angles TS′T′, TST′.

But $\angle$TST′ = ½ $\angle$BSB′, and $\angle$TS′T′ = ½ external angle BS′B′ = $\pi - \frac{1}{2}$BSB′;

therefore angle at which the circles cut $= \pi -$ BSB′ $= 2\tan^{-1}\dfrac{c}{b}$.

The first part of this problem I have already set in the form, "If a circle pass through the foci of an ellipse, and common tangents be drawn to the ellipse and circle, their points of contact with the circle will lie on the tangents to the ellipse at the ends of the minor axis."

2780. (Proposed by Professor HIRST.)—The envelope of the chord common to an ellipse and its circle of curvature is a curve of the fourth class which has three double tangents, one at infinity and the two others coincident with the conjugate diameters equally inclined to the axis. Prove this, examine the curve, and consider the cases of the hyperbola and parabola.

Solution by J. J. WALKER, M.A.

The equation to the ellipse being $b^2x^2+a^2y^2=a^2b^2$, if the coordinates of any point on it be $a\cos\theta$, $b\sin\theta$, the reciprocals of the intercepts made on the axes by the chord common to the ellipse and the circle of curvature at this point will be $\alpha = \dfrac{\cos\theta}{a\cos 2\theta}$, $\beta = \dfrac{-\sin\theta}{b\cos 2\theta}$, and eliminating θ between these equations, $a^2\alpha^2+b^2\beta^2=(a^2\alpha^2-b^2\beta^2)^2$ is the envelope of the chord,—a curve of the fourth class, having as double tangents the lines joining the points $a\alpha \pm b\beta = 0$ with the centre,—*i. e.*, the equal conjugate diameters; also the line $\alpha=0$, $\beta=0$.

For the hyperbola, the sign of b^2 is to be changed, and the first pair of double tangents become imaginary.

In the case of the parabola $y^2=4mx$, the tangential equation to the envelope of the chord common to it and its circle of curvature at any point

is similarly found to be $\alpha = 3m\beta^2$, a parabola having the same vertex with the given parabola, and as its axis the production of axis of the other. Its parameter is three times that of the given parabola.

The equations to the above envelopes in Cartesian coordinates are

$$(b^2x^2+a^2y^2-4a^2b^2)^3+27\,(b^2x^2-a^2y^2)=0 \quad \text{and} \quad y^2+12mx=0$$

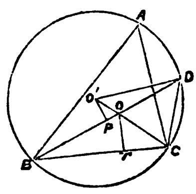
respectively, the equations to the ellipse and parabola being $b^2x^2+a^2y^2=a^2b^2$ and $y^2=4mx$ respectively. From the former it appears that the points of contact of the equi-conjugate diameters lie on a similar and coaxal ellipse, and that there are cusps at these points.

[The equation of the envelope has been given by Mr. WOLSTENHOLME, in his Solution of Question 2579 (2), (*Reprint*, Vol. X. p. 91) in the neat form

$$\left(\frac{x}{a}+\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{x}{a}-\frac{y}{b}\right)^{\frac{2}{3}}=2,$$

which has cusps at the points $\dfrac{x^2}{a^2}=\dfrac{y^2}{b^2}=2$, and of course two at infinity. The form of the equation to the envelope given by Mr. WALKER in Question 2834 may be readily deduced from Mr. WOLSTENHOLME'S.]

To express the distance between the centres of the circumscribed and inscribed circles in terms of the radii of those circles. By J. WALMSLEY, B.A.



Let O', O be centres of circumscribed and inscribed circles respectively of the triangle ABC. Join $O'O$, OB, OC. Produce BO to D, and join DO'. Draw $O'P$ perpendicular to BD, and therefore bisecting it.

Then $$BO=\frac{r}{\sin\frac{1}{2}B},$$

and $$DCO=DCA+ACO=DBA+ACO$$
$$=OBC+OCB=DOC;$$

therefore $$DO=DC=\frac{a\sin\frac{1}{2}B}{\sin A}=2R\sin\tfrac{1}{2}B.$$

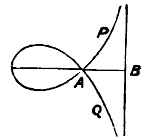
Hence $$O'D^2-O'O^2=DP^2-OP^2=DO\,.\,OB=2Rr.$$

therefore $$O'O^2=R^2-2Rr.$$

In an analogous manner, if O_1 be centre and r_1 radius of escribed circle touching BC, we may show that $O'O_1{}^2=R^2+2Rr_1$.

2776. (Proposed by W. K. CLIFFORD, B.A.) — Through A, the double point of a circular cubic, draw AB perpendicular to the asymptote; if chords be drawn to the curve subtending a right angle at the double point, show that there is a fixed point in AB at which also they subtend a right angle.

Solution by the REV. J. WOLSTENHOLME, M.A.



The equation may be taken $y^2 = x^2 \dfrac{x+a}{b-x}$, the origin being the double point. If (x_1y_1), (x_2y_2) be the ends of two chords AP, AQ at right angles to each other, $x_1x_2 + y_1y_2 = 0$; and

$$y_1^2 = x_1^2 \frac{x_1+a}{b-x_1}, \quad y_2^2 = x_2^2 \frac{x_2+a}{b-x_2};$$

therefore $(x_1+a)(x_2+a) = (b-x_1)(b-x_2)$, therefore $x_1 + x_2 = b - a$.

But the equation of the circle on PQ is

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0, \text{ or } x^2+y^2-x(x_1+x_2)-y(y_1+y_2) = 0;$$

therefore, when $y = 0$, $x = 0$ or $x = x_1 + x_2 = b - a$,

giving a fixed point on AB at which PQ subtends a right angle.

2544. (Proposed by Professor BALL.)—There are two values of d for which α, β, γ, the roots of the cubic $ax^3 + 3bx^2 + 3cx + d = 0$ fulfil the linear relation $A\alpha + B\beta + C\gamma = 0$. Show how to find the quadratic equation of which these values of d are the roots.

I. *Solution by the* REV. J. WOLSTENHOLME, M.A.

The relations $\alpha+\beta+\gamma = -\dfrac{3b}{a}$, $A\alpha + B\beta + C\gamma = 0$, $\beta\gamma + \gamma\alpha + \alpha\beta = \dfrac{3c}{a}$,

lead at once to a quadratic equation in γ; whence

$$\gamma_1 + \gamma_2 = -\frac{3b}{a}\,\frac{A^2+B^2-C(A+B)}{A^2+B^2+C^2-BC-CA-AB} \equiv 2m,$$

$$\gamma_1\gamma_2 = \frac{3ac(A-B)^2 + 9b^2AB}{a^2(A^2+\ldots-BC-\ldots)} \equiv n.$$

But $\quad -(d_1+d_2) = +a(\gamma_1^3+\gamma_2^3) + 3b(\gamma_1^2+\gamma_2^2) + 3c(\gamma_1+\gamma_2)$,

and $\quad d_1d_2 = \gamma_1\gamma_2(a\gamma_1^2 + 3b\gamma_1 + 3c)(a\gamma_2^2 + 3b\gamma_2 + 3c)$,

which can be immediately expressed in terms of m and n.

II. *Solution by* J. J. WALKER, M.A.

By the terms of the question,

$$B\beta + C\gamma = -A\alpha \ \ldots\ldots (1), \quad a\beta + a\gamma = -a\alpha - 3b \ \ldots\ldots (2),$$

$$a\beta\gamma + a(\beta+\gamma)\alpha = 3c \ \ldots\ldots (3).$$

From (1) and (2), $a(B-C)\beta = a(C-A)\alpha + 3bC \ \ldots\ldots (4)$,

$$a(C-B)\gamma = a(B-A)\alpha + 3bB \ \ldots\ldots (5);$$

$$-a^2(B-C)^2\beta\gamma = a^2(A-B)(A-C)\alpha^2+3ab\left\{2BC-A(B+C)\right\}\alpha+9b^2BC \quad \ldots\ldots\ldots\ldots(6).$$

Eliminating β and γ from (2), (3), (6), there results

$$a^2\left\{(A-B)(A-C)-(B-C)^2\right\}\alpha^2+3ab\left\{B^2+C^2-A(B+C)\right\}\alpha$$
$$+3\left\{ac(B-C)^2+3b^2BC\right\}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(7);$$

also

$$a\alpha^3+3b\alpha^2+3c\alpha+d=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(8).$$

The resultant of (7) and (8) is

$$a'^4d^2+9a'\left\{(3b'^2-2a'c')(a'b-ab')-a'b'(a'c-ac')\right\}d$$
$$+81c'^2(a'b-ab')^2+27c'(a'c-ac')\left\{a'(a'c-ac')-3b'(a'b-ab')\right\}=0 \ldots (9),$$

if we write

$$a'=a^2\left\{(A-B)(A-C)+(B-C)^2\right\},$$
$$b'=ab\left\{B^2+C^2-A(B+C)\right\},$$
$$c'=ac(B-C)+3b^2BC,$$

whence $a'b-ab'=a^2b(A^2+BC)$, $a'c-ac'=a\left\{ac(A-B)(A-C)-3b^2BC\right\}$.

Making these substitutions in (9), it becomes divisible by a^4, and is the required quadratic for d. The coefficients are not capable of any important reduction, as far as I have examined them.

1587. (Proposed by the Rev. T. P. Kirkman, F.R.S.)—Apollo and the Muses accepted the challenge of Jove, to vary the arrangement of themselves on their fixed and burnished couches at his evening banquets, till every three of them should have occupied, once and once only, every three of the couches, in every and any order. In how many days, and how many ways, did they accomplish the feat, keeping one arrangement of themselves through all the solutions? Required two or more of these solutions, clearly indicated, so as to save space, by cyclical operations.

Solution by the Proposer.

The solution is the positive half of the group of 10.9.8.2 made with 10 elements, which was first given by me in the *Memoirs of the Literary and Philosophical Society of Manchester*, 1861, and whose *title* I gave thus in their *Proceedings* five or six years ago:—

$$10.9.8.2 = 1+180_{82}+144_{5^2}+45_{2^41^2}+80_{3^31}+270_{4^21^2}$$
$$+30_{2^31^4}+180_{81^2}+144_{10}+90_{4^22}+240_{631}+36_{2^5}:\ Q=2520.$$

The positive half is the first six terms on the right, comprising, besides unity, the natural order, 180 substitutions having each a circle of 8 and a circle of 2, 144 substitutions having each two circles of 5, 45 having each four

circles of 2, and 2 elements undisturbed, 80 having three circles of 3 and one undisturbed element, &c.

This group of 10.9.8.2, as well as every group so given by its title, can be easily constructed from its title; but it is impossible to state the rules here. It is enough to give one of the 2520 equivalent groups of 10.9.8, thus,

1234567890	1234567890	1234567890
+0324897561	3215648970	1473692580
+3017594862	3126459780	1327984650
+2107864593	4567891230	1742853960
+6280715394	5648972310	1685739240
+3814069275	6459783120	1956278430
+8243907156	7891234560	1869425370
+8639520147	8972315640	1598346720
+4261539078	9783126450	
+2165437809		

under the form (A);G;H where G;H is a group of 9.8, the product of two groups G and H. (A) is no group. The 10.9.8 cannot be written as the product of so few as three groups. We have the group by adding to G;H its nine derivatives by the substitutions of (A). If these be $1, A_1, A_2, \ldots A_9$; $1, g_1, g_2, \ldots g_8$ those of G; and $1 h_1 h_2 \ldots h_7$ those of H; the group

$$(A);G;H = G;H + A_1;G;H + A_2;G;H + \ldots + A_9;G;H,$$

will be found to stand the test, that every product of its substitutions, if $1 = A_0 = g_0 = h_0$, is of the form $A_p;g_q;h_r$. Neither the algebraic nor the tactical demonstration of this can find space here. The group proves itself. These 10.9.8 arrangements are one of the 2520 solutions of the problem which Apollo's party dined out day by day for nearly 5000 years, at some expense of Jove's nectar. There are three, and only three, solutions which have in common the substitutions of G, (A);G;H, (A′);G;H′, and (A″);G;H″. No two solutions have more than eight arrangements in common besides unity. The above group (A);G;H is necessary as to the number, and sufficient as to the form, of its arrangements: necessary, because Apollo, Clio, and Urania cannot seat themselves in every way in fewer than 10.9.8 arrangements; and sufficient, because if in Θ and Φ, any pair of arrangements K, L, and M were seated alike, the group would contain the substitution $\frac{\Theta}{\Phi}$, which has KLM undisturbed; but the positive title has no substitution showing three elements undisturbed. From (A);G;H once written out can be formed all other solutions, by the operation $\theta;(A);G;H;\theta^{-1}$, where θ is any substitution not before used. Thus, $\theta = \theta^{-1}$ may be 2134567890 for a second solution.

NOTE.—The unfashionable reader who wishes to learn how to find all possible groups made with n elements, to determine their titles and the number of their equivalents, and from the titles to construct the groups by a direct tactical method, without troubling himself with congruences, has only to fish up, if he can, from a basket in Manchester, a Memoir of mine which has lain there for some years. Our learned societies agree with me in the opinion that it is absurd to waste English paper on such upstart subjects as Groups and Polyedra, which are fit for nothing but make-believe prize-questions of the French Academy. They have done quite enough in placing it beyond dispute in their Proceedings that I have thoroughly discussed both these vast theories, as difficult as they are unvalued.

2847. (Proposed by M. W. CROFTON, F.R.S.)—Prove that

$$\epsilon^{hD^2}.\,\epsilon^{-kx^2} = (1+4hk)^{-\frac{1}{2}}\epsilon^{-\frac{kx^2}{1+4hk}}, \text{ where } D = \frac{d}{dx}.$$

Solution by J. J. WALKER, M.A.

Expanding the symbolical operator,

$$\epsilon^{hD^2}.\,\epsilon^{-kx^2} = \left(1+\frac{hD^2}{1}+\frac{h^2D^4}{1.2}+\dots\right)\epsilon^{-kx^2}.$$

But by the known formula for the successive differential coefficients of ϵ^{-kx^2} with respect to x,

$$\begin{aligned}
\epsilon^{kx^2}\,D^0.\,\epsilon^{-kx^2} &= 1,\\
\epsilon^{kx^2}h\,D^2.\,\epsilon^{-kx^2} &= -\frac{2.1}{1}hk + hk^2(2x)^2,\\
\epsilon^{kx^2}h^2D^4.\,\epsilon^{-kx^2} &= \frac{4\dots1}{(1.2)^2}h^2k^2 - \frac{4.3}{1.2}h^2k^3(2x)^2 + \frac{h^2k^4}{1.2}(2x)^4,\\
\epsilon^{kx^2}h^3D^6.\,\epsilon^{-kx^2} &= -\frac{6\dots1}{(1.2.3)^2}h^3k^3 + \frac{6\dots3}{(1.2)^2.3}h^3k^4(2x)^2\\
&\quad - \frac{6.5}{1.2.3}h^3k^5(2x)^4 + \frac{h^3k^6}{1.2.3}(2x)^6,\\
\dots\dots &= \dots\dots\dots\dots
\end{aligned}$$

It will be found on examination that the rth term (or line) of the pth column in the sum of the above developments is equal to

$$(-1)^{r-1}\frac{(4hk^2x^2)^{p-1}}{1.2\dots(p-1)} \times \frac{(2r+2p-4).(2r+2p-5)\dots(2p-1).(4hk)^{r-1}}{1.2\dots(r-1).r\dots(r+p-2).p.(p+1)\dots(r-1).(2)^{2r-2}},$$

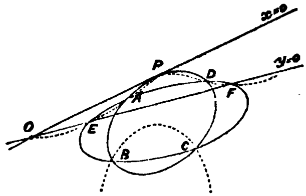
when r is greater than p; but when r is equal to or less than p, the denominator of the last factor is $1.2\dots(r-1).p.(p+1)\dots(p+r-2).2^{2r-2}$. In both cases this factor is equal to $\dfrac{(2p-1)(2p+1)\dots(2p+2r-5)}{1.2\dots(r-1).2^{r-1}}(4hk)^{r-1}$, which multiplied by $(-1)^{r-1}$ is the rth term in the development of $(1+4hk)^{-\frac{2p-1}{2}}$. From this it results that $\epsilon^{kx^2}.\,\epsilon^{hD^2}.\,\epsilon^{-kx^2} = (1+4hk)^{-\frac{1}{2}}\epsilon^{\frac{4hk^2x^2}{1+4hk}}$; and multiplying both sides of this by ϵ^{-kx^2}, there results for $\epsilon^{hD^2}.\,\epsilon^{-kx^2}$ the value given in the Question.

[This theorem, which the Proposer has found to possess important applications in the Theory of Errors of Observation, seems deducible in the most direct, though not the most elementary, manner, by putting ϵ^{-kx^2} for $\phi(x)$ in POISSON's famous transformation (*Traité de Mécanique*, Tom. II., p. 356), which gives

$$\epsilon^{hD^2}\phi(x) = \frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty}\epsilon^{-\omega^2}\phi(x+2h^{\frac{1}{2}}\omega)\,d\omega.]$$

2623. (Proposed by W. S. McCay, B.A.)—If a conic pass through the four points of contact of tangents to a cubic from a point (A) on the curve, and through two other points (B, C) on the cubic; then A is the pole of BC with regard to the conic.

Solution by F. D. Thomson, M.A.



The equation to any cubic may be written

$$S \equiv xV + kyU = 0 \ldots\ldots (1),$$

where U, V are conics intersecting in four points A, B, C, D on the curve, and x and y are straight lines intersecting on the curve, in the point O suppose.

The equation to the polar conic of (1) with reference to the point (x', y', z') is

$$x'\frac{dS}{dx} + y'\frac{dS}{dy} + z'\frac{dS}{dz} = 0,$$

which may be written

$$x'V + ky'U + xP_v + kyP_u = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where P_u denotes the polar of (x', y', z') with respect to U. But if (x', y', z') be on the line OP which touches U, $x' = 0$, and P_u becomes $x\dfrac{du}{dx'}$.

Hence the equation (2) reduces to

$$ky'U + x\left(P_v + ky\frac{du}{dx'}\right) = 0.$$

Therefore, if U is itself the polar conic of the point P,

$$P_v + ky\frac{du}{dx'} = 0 \text{ identically.}$$

Therefore P_v coincides with $y = 0$, *i.e.*, with EF, or EF is the polar of P with respect to V.

2606. (Proposed by C. W. Merrifield, F.R.S.)—The developable circumscribing two surfaces of the second degree touches either of them along a curve, which is its intersection with another surface of the second degree.

I. *Solution by the* Rev. R. Townsend, F.R.S.

If U and V be the two quadrics, U′ the polar reciprocal of U with respect to V, and V′ the polar reciprocal of V with respect to U; then, for every plane tangent to both U and V, the point of contact with U being a point on V′, and the point of contact with V being a point on U′, therefore, &c. (See Salmon's *Geometry of Three Dimensions*, 2nd Ed., Art. 207.)

If $a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 = 0$ and $a'\alpha^2 + b'\beta^2 + c'\gamma^2 + d'\delta^2 = 0$ be the equations of U and V referred to their common self-reciprocal tetrahedron, then are

$$\frac{a'^2}{a}\alpha^2+\frac{b'^2}{b}\beta^2+\frac{c'^2}{c}\gamma^2+\frac{d'^2}{d}\delta^2=0 \quad\text{and}\quad \frac{a^2}{a'}\alpha^2+\frac{b^2}{b'}\beta^2+\frac{c^2}{c'}\gamma^2+\frac{d^2}{d'}\delta^2=0$$

those of U′ and V′ referred to the same tetrahedron: which, as is otherwise evident geometrically, is consequently self-reciprocal with respect to the whole four surfaces at once. (See same, Art. 206.)

II. *Solution by the* Proposer.

Referring the two surfaces to the centre of one of them, and to axes parallel to conjugate diameters in both, their equations may be written thus:—

$$U = \alpha_1 x_1^2+\beta_1 y_1^2+\gamma_1 z_1^2+\delta_1 = 0,$$

$$V = \alpha_2(x_2-l)^2+\beta_2(y_2-m)^2+\gamma_2(z_2-n)^2+\delta_2-\alpha_2 l^2-\beta_2 m^2-\gamma_2 n^2 = 0.$$

The tangent plane of U is $\alpha_1 x_1 x+\beta_1 y_1 y+\gamma_1 z_1 z+\delta_1 = 0.$

The tangent plane of V is

$$\alpha_2(x_2-l)(x-l)+\beta_2(y_2-m)(y-m)+\gamma_2(z_2-n)(z-n)$$
$$+\delta_2-\alpha_2 l^2-\beta_2 m^2-\gamma_2 n^2 = 0,$$

or $\alpha_2(x-l)x+\beta_2(y_2-m)y+\gamma_2(z_2-n)z+\delta_2-\alpha_2 lx_2-\beta_2 my_2-\gamma_2 mz_2 = 0.$

And, if these are to be identical,

$$\frac{\alpha_1 x_1}{\alpha_2(x_2-l)}=\frac{\beta_1 y_1}{\beta_2(y_2-m)}=\frac{\gamma_1 z_1}{\gamma_2(z_2-n)}=\frac{\delta_1}{\delta_2-\alpha_2(x_2-\beta_2 my_2-\gamma_2 mz_2)}.$$

Substituting these values of $x_1 y_1 z_1$ in U, we get

$$\frac{\alpha_2^2\delta_1(x_2-l)^2}{\alpha_1}+\frac{\beta_2^2\delta_1(y_2-m)^2}{\beta_1}+\frac{\gamma_2^2\delta_1(z_2-n)^2}{\gamma_1}$$
$$+(\delta_2-\alpha_2 lx_2-\beta_2 my_2-\gamma_2 nz_2)^2 = 0.$$

Hence the developable of two quadrics touches each along a curve through which another quadric can be drawn.

Conversely, the envelope of the tangent planes to a quadric, along its intersection with another quadric, is the developable circumscribing the first and some other quadric.

2804. (Proposed by S. Tebay.)—A straight pole stands vertically on a slope inclined to the south. If it be broken at random by the wind blowing in a given direction, so that the upper end of the pole rests upon the slope, determine the probable area of the triangle thus formed; and deduce the result for a horizontal plane.

Solution by the Proposer.

Let a be the length of the pole, x the length of the portion standing, α the inclination of the slope to the vertical, β the azimuth of the wind measured on the slope, and ϕ the angle between the standing piece and the base of the triangle. Then $\cos\phi = \cos\alpha\cos\beta$, and

$$\Delta = \tfrac{1}{2}x\sin\phi\left\{x\cos\phi \mp (a^2-2ax+x^2\cos^2\phi)^{\frac{1}{2}}\right\}.$$

The greatest value of x is $\dfrac{a}{1+\sin\phi}$, which is also a measure of the number of triangles. Hence we find

$$\int\Delta\,dx = \tfrac{1}{2}a^3\left\{\frac{\sin\phi\cos\phi}{3(1+\sin\phi)^3} - \frac{\sin\phi}{3\cos^2\phi} + \frac{\sin\phi}{2\cos^4\phi} - \frac{\sin^3\phi}{2\cos^5\phi}\log\cot\tfrac{1}{2}\phi\right\};$$

and therefore the average area of the triangle is

$$\tfrac{1}{2}a^2\sin\phi\,(1+\sin\phi)\left\{\frac{\cos\phi}{3(1+\sin\phi)^3} - \frac{1}{3\cos^3\phi} + \frac{1}{2\cos^4\phi} - \frac{\sin^2\phi}{2\cos^5\phi}\log\cot\tfrac{1}{2}\phi\right\}.$$

When $a=\frac{1}{2}\pi$, we have $\phi=\frac{1}{2}\pi$; and expanding $\log\cot\frac{1}{2}\phi = \frac{1}{2}\log\dfrac{1+\cos\phi}{1-\cos\phi}$, we find the average area $=\frac{1}{18}a^2$.

2618. (Proposed by the Rev. N. M. Ferrers, M.A.)—A pack of n different cards is laid, face downwards, on a table. A person names a certain card, that and all the cards above it are shown to him and removed; he names another, and the process is repeated. Prove that the chance of his naming the top card during the operation is

$$1-\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}-\frac{1}{\underline{|4}}+\ldots\ldots-\frac{(-1)^n}{\underline{|n}}.$$

Solution by Professor Whitworth.

Let P_n represent the chance of the top card being named in the series of operations on n cards.

At the first operation any of the n cards is equally likely to be named.

If the top one is named, what is required is done, and the *à posteriori* chance of success is unity.

If the second be named, there are $n-2$ cards left, and the *à posteriori* chance of success is P_{n-2}.

If the third be named, then the *à posteriori* chance of success is P_{n-3}, and so on.

If the last card be named, the chance is zero; therefore

$$P_n = \frac{1}{n}\left\{1+P_{n-2}+P_{n-3}+\ldots\ldots+P_2+P_1\right\},$$

or
$$nP_n = 1+P_1+P_2+\ldots\ldots+P_{n-3}+P_{n-2}.$$

Similarly (writing $n-1$ for n)

$$(n-1)P_{n-1} = 1+P_1+P_2+\ldots\ldots+P_{n-3};$$

therefore, by subtraction, $nP_n-(n-1)P_{n-1} = P_{n-2}$,

or
$$n(P_n-P_{n-1}) = -(P_{n-1}-P_{n-2}).$$

Multiply both sides by $(-1)^n \lfloor n-1$, then

$$(-1)^n \lfloor n \, (P_n - P_{n-1}) = (-1)^{n-1} \lfloor n-1 \, (P_{n-1} - P_{n-2}).$$

Now, writing $n-1$, $n-2$, $n-3$, &c. successively for n, we obtain, *ex equali*,

$$(-1)^n \lfloor n \, (P_n - P_{n-1}) = \lfloor 2 \, (P_2 - P_1).$$

But it is obvious that $P_1 = 1$ and $P_2 = \frac{1}{2}$, therefore

$$(-1)^n \lfloor n \, (P_n - P_{n-1}) = -1, \quad \text{or} \quad P_n - P_{n-1} = \frac{(-1)^{n-1}}{\lfloor n}.$$

Writing 2, 3, 4, &c. successively for n, we obtain

$$P_2 - P_1 = -\frac{1}{\lfloor 2}, \quad P_3 - P_2 = +\frac{1}{\lfloor 3}, \quad P_4 - P_3 = -\frac{1}{\lfloor 4}, \quad \text{\&c.}$$

$$P_n - P_{n-1} = \frac{(-1)^{n-1}}{\lfloor n}.$$

Therefore, by addition,

$$P_n - P_1 = -\frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} - \frac{1}{\lfloor 4} + \dots - \frac{(-1)^n}{\lfloor n},$$

or

$$P_n = 1 - \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} - \frac{1}{\lfloor 4} + \dots - \frac{(-1)^n}{\lfloor n}.$$

With the notation of Questions 2637, 2648, this chance (of naming a top card) is $1 - e_n^{-1}$, and the chance of not naming a top card is e_n^{-1}.

2813. (Proposed by J. J. WALKER, M.A.)—The equation of a conic referred to an axis and tangent at vertex being $ax^2 + by^2 + 2dx = 0$, if the conic be turned about the vertex in its own plane through a right angle, the locus of the intersection of any tangent in the original position with the same line in the new position of the conic is the (bicircular) quartic

$$b\left\{a(x^2+y^2) + 2d(x+y)\right\}(x^2+y^2) = d^2(x-y)^2,$$

and the corresponding locus for the normal is the sextic

$$b\left\{a(x^2+y^2) + b(x-y)^2\right\}\left\{a(x^2+y^2) + d(x+y)\right\}^2 = (a-b)^2 d^2 (x^2-y^2)^2.$$

What does this latter equation become in the case of the parabola?

I. *Quaternion Solution by* W. H. LAVERTY, B.A.

1. Taking for simplicity the equation to the conic in the form

$$b^2x^2 + a^2y^2 = 2ab^2x,$$

the vector equations to the tangents are

$$\rho = \alpha + \alpha\cos\tau + \beta\sin\tau + x(-\alpha\sin\tau + \beta\cos\tau),$$

$$\rho = \frac{a}{b}\beta(1+\cos\tau) - \frac{b}{a}\alpha\sin\tau + y\left(\frac{a}{b}\beta\sin\tau + \frac{b}{a}\alpha\cos\tau\right);$$

and that these may coincide, we must have

$$x(a^2\sin^2\tau + b^2\cos^2\tau) = a(a\sin\tau + b\cos\tau) + (a^2-b^2)\sin\tau\cos\tau + ab;$$

whence substituting, and putting $a^2\sin^2\tau + b^2\cos^2\tau = \mathrm{P}$, we have for the required locus

$$\rho\,.\,\mathrm{P} = \left\{b\alpha(b\cos\tau - a\sin\tau) + a\beta(b\cos\tau + a\sin\tau)\right\}(1+\cos\tau).$$

To transform this to a Cartesian equation, we have

$$\frac{x\,.\,\mathrm{P}}{ab(1+\cos\tau)} = b\cos\tau - a\sin\tau, \quad \text{and} \quad \frac{y\,.\,\mathrm{P}}{ab(1+\cos\tau)} = b\cos\tau + a\sin\tau;$$

therefore $$x^2+y^2 = \frac{2a^2b^2(1+\cos\tau)^2}{\mathrm{P}} \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

also $$(x-y)^2\mathrm{P}^2 = 4a^2b^2(1+\cos\tau)^2a^2\sin^2\tau$$

or $$(x-y)^2\mathrm{P} = 2a^2\sin^2\tau(x^2+y^2) \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and $$(x+y)\,\mathrm{P} = 2ab(1+\cos\tau)\,b\cos\tau;$$

from (1), $$\left\{x^2+y^2-2a(x+y)\right\}\mathrm{P} = 2a^2b^2\left\{(1+\cos\tau)^2 - 2\cos\tau(1+\cos\tau)\right\}$$
$$= 2a^2b^2\sin^2\tau;$$

from (2), $$\left\{x^2+y^2-2a(x+y)\right\}(x^2+y^2) = b^2\frac{2a^2\sin^2\tau(x^2+y^2)}{\mathrm{P}} = b^2(x-y)^2,$$

or $$b\left\{a(x^2+y^2)+2d(x+y)\right\}(x^2+y^2) = d^2(x-y)^2.$$

2. The vector equations to the normals are

$$\rho = \alpha + \alpha\cos\tau + \beta\sin\tau + x(b^2\alpha\cos\tau + a^2\beta\sin\tau),$$
$$\rho' = \frac{a}{b}\beta + \frac{a}{b}\beta\cos\tau - \frac{b}{a}\alpha\sin\tau + y(ba\beta\cos\tau - ab\alpha\sin\tau);$$

and that these may coincide, we have

$$-abx(a^2\sin^2\tau + b^2\cos^2\tau) = a\left\{b\cos\tau - a\sin\tau\right\} + ab + (b^2-a^2)\cos\tau\sin\tau;$$

whence substituting, and putting

$$a^2\sin^2\tau + b^2\cos^2\tau = \mathrm{P} \quad \text{and} \quad a^2(1+\cos\tau) - b^2\cos\tau = \mathrm{Q},$$

we get for the required locus

$$ab\mathrm{P}\rho = \sin\tau\,\mathrm{Q}\left\{b\alpha(a\sin\tau + b\cos\tau) - \beta a(b\cos\tau - a\sin\tau)\right\}.$$

To transform this to a Cartesian equation, we have

$$x\mathrm{P} = \sin\tau\mathrm{Q}(a\sin\tau + b\cos\tau), \qquad y\mathrm{P} = \sin\tau\mathrm{Q}(a\sin\tau - b\cos\tau);$$

whence we easily find that

$$\frac{b^2(x+y)^2 + a^2(x-y)^2}{(x^2-y^2)^2}\left\{x^2+y^2-a(x+y)\right\}^2$$
$$= \frac{4a^2b^2\sin^2\tau\,\mathrm{Q}^2}{\mathrm{P}^2}\;\frac{\mathrm{P}^4}{a^2b^2\,16\sin^6\tau\,\mathrm{Q}^4\cos^2\tau}\;\frac{4\sin^4\tau\,\mathrm{Q}^2(a^2-b^2)^2\cos^2\tau}{\mathrm{P}^2} = (a^2-b^2)^2,$$

or $$b\left\{a(x+y)^2 + b(x-y)^2\right\}\left\{a(x^2+y^2)+d(x+y)\right\}^2 = (a-b)^2d^2(x^2-y^2)^2.$$

3. In the parabola our equations become

$$\mathrm{P}x = t\mathrm{Q}\left\{at-b\right\} \qquad \text{and} \qquad \mathrm{P}y = t\mathrm{Q}\left\{at+b\right\},$$

where P and Q now equal respectively $a^2t^2+b^2$ and $\frac{1}{2}a^2t^2+b^2$.

Let $\tau^2 = b\,t^2 = \frac{1}{4}alt^2$ (where l is the latus rectum); then we have easily

$$xRl = \tau(2\tau - l)\,S \quad \text{and} \quad yRl = \tau(2\tau + l)\,S,$$

where $R = 4\tau^2 + l^2$, and $S = 2\tau^2 + l^2$;

therefore $$l\frac{x^2+y^2}{x+y} = \frac{S}{2}, \text{ and } -2\tau = l\frac{x+y}{x-y};$$

therefore $$4\frac{x^2+y^2}{x+y} = l\frac{(x+y)^2+2(x-y)^2}{(x-y)^2}.$$

II. *Solution by the* PROPOSER; R. TUCKER, M.A.; *and others.*

1. The tangent being $(ax'+d)\,x + by'y + dx' = 0$, the equation to the same line, when the conic is turned through a right angle, will be

$$by'x - (ax'+d)\,y - dx' = 0.$$

Adding these equations, we get $(ax'+d)(x-y) = -by'(x+y)$;

and eliminating y' by means of the equation to the conic,

$$(ax'+d)^2(x-y)^2 + b(ax'^2+2dx')(x+y)^2 = 0 \;\ldots\ldots\ldots\; (1).$$

Again, eliminating y' between the two equations above, we have

$$(ax'+d)(x^2+y^2) + dx'(x+y) = 0 \;\ldots\ldots\ldots\ldots\; (2).$$

The elimination of x' between (1) and (2) gives the first locus, which in the case of the parabola evidently reduces to a (circular) cubic.

2. The equations to the normal in the original and new position of the conic are, respectively,

$$by'x - (ax'+d)\,y + y'\left\{(a-b)\,x' + d\right\} = 0,$$

and $$(ax'+d)\,x + by'y + y'\left\{(a-b)\,x' + d\right\} = 0.$$

Pursuing steps similar to those in the case of the tangent, the required locus is easily obtained. In the case of the parabola, developing, reducing, dividing by a, and finally making $a=0$, there results

$$2(x-y)^2\left\{b(x^2+y^2) + d(x+y)\right\} + d(x+y)^3 = 0.$$

If the given conic be a circle, the locus evidently becomes a circle also.

2839. (Proposed by J. J. WALKER, M.A.)—In a triangle the bisector of the base is equal to the less side and also to one-half of the greater side; determine the three angles.

Solution by the PROPOSER; R. TUCKER, M.A.; *and others.*

Let D be the middle point of the base BC, A the vertex of the triangle, and AC = AD = ½AB. Then AC must bisect the angle between AD and BA produced (Euc. VI. A), and ∠ CAD = 180° − ∠ BAC,

therefore $\angle \mathrm{BAC} = 2\mathrm{C}$, therefore $3\mathrm{C} = 180^\circ - \mathrm{B}$,

whence $$\sin 3\mathrm{C} = 3\sin \mathrm{C} - 4\sin^3 \mathrm{C} = \sin \mathrm{B}.$$

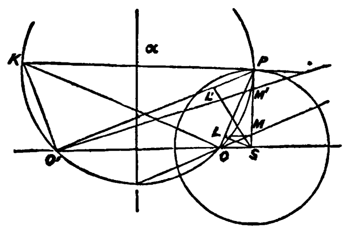
Again, $\sin \mathrm{C} : \sin \mathrm{B} = \mathrm{AB} : \mathrm{AC} = 2 : 1$, or $\sin \mathrm{C} = 2 \sin \mathrm{B}$,

whence $$6 \sin \mathrm{C} - 8 \sin^3 \mathrm{C} = \sin \mathrm{C}.$$

Disregarding the irrelevant solution $\mathrm{C}=0$ or 180°, we have $\sin^2 \mathrm{C} = \frac{5}{8}$.
But $\cos \mathrm{BAC} = 1 - 2\sin^2 \mathrm{C} = -\frac{1}{4}$, and $\sin^2 \mathrm{B} = \frac{1}{4} \sin^2 \mathrm{C} = \frac{5}{32}$.

2818. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—Given a circle S and a straight line α not meeting S in real points; O, O′ are the two point-circles to which, and S, α is the radical axis; two conics are drawn osculating S in the same point P, and having one focus at O, O′ respectively; prove that the corresponding directrices coincide.

Solution by R. TUCKER, M.A.



Join OP, O′P, and draw OK, O′K, perpendiculars to them; then it is plain that these lines intersect in K on the circle through O′OP, cutting S orthogonally; and K is the intersection of the directrices. From the centre S draw SL, SL′ perpendiculars on OP, O′P, and LM, L′M′ perpendiculars on SP; then (Salmon's *Conics*, Art. 243) OM, O′M′ are the directions of the axes of the conics, and

$$\mathrm{SM} = \mathrm{SP} \sin^2 \mathrm{OPS}, \quad \mathrm{SM}' = \mathrm{SP} \sin^2 \mathrm{O'PS};$$

therefore $$\frac{\mathrm{SM}}{\mathrm{SM}'} = \frac{\sin^2 \mathrm{OPS}}{\sin^2 \mathrm{O'PS}} = \frac{\mathrm{OP}^2}{\mathrm{O'P}^2} = \frac{\mathrm{SO}}{\mathrm{SO}'};$$

that is, OM is parallel to O′M′, and the directrices (the perpendiculars from K on the axes) coincide.

2761. (Proposed by T. COTTERILL, M.A.)—Find the conic envelope of the radical axis of two circles, one of which is fixed, whilst the other passes through a fixed point. If the radii of the circles are equal, find the foci and axes of the envelope.

Solution by the Rev. J. WOLSTENHOLME, M.A.

Let the fixed point be origin, $x^2+y^2-2\lambda x-2\mu y = 0$ the equation of the moving circle, and $(x-a)^2+y^2 = r^2$ that of the fixed circle; then, if r' be the given radius of the moving circle, the radical axis is

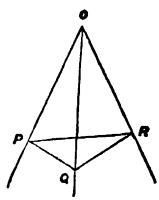
$$\lambda x + \mu y - ax + \tfrac{1}{2}(a^2 - r^2) = 0, \text{ with the condition } \lambda^2 + \mu^2 = r'^2;$$

and for the envelope $\dfrac{x}{\lambda} = \dfrac{y}{\mu} = \dfrac{ax - \frac{1}{2}(a^2 - r^2)}{r'^2} = \dfrac{(x^2+y^2)^{\frac{1}{2}}}{r'}$,

or the envelope is a conic having its focus at the fixed point, its directrix bisecting the tangents drawn from the fixed point to the fixed circle, and eccentricity the ratio of the distance between the fixed point and the centre of the fixed circle to the radius of the moving circle. If the radii be equal, the envelope is the reciprocal polar of the fixed circle with respect to a circle whose centre is the fixed point and radius $\left\{\frac{1}{2}(a^2 - r^2)\right\}^{\frac{1}{2}}$.

2724. (Proposed by S. TEBAY, B.A.)—Three smooth rings, P, Q, R, are thread on an endless string of given length, and constrained to move on three straight rods, OP, OQ, OR. To investigate the motion corresponding to a slight arbitrary disturbance of the system.

Solution by the PROPOSER.



Let $\quad OP = x, \quad OQ = y, \quad OR = z,$

$\quad PQ = u, \quad OR = v, \quad RP = w,$

$\delta_1, \delta_2, \delta_3$ the inclinations of OP, OQ, OR to the vertical; and T the tension of the string;

$$POQ = \alpha, \quad OPQ = \phi, \quad OPR = \phi',$$
$$QOR = \beta, \quad OQR = \chi, \quad OQP = \chi',$$
$$ROP = \gamma, \quad ORP = \psi, \quad ORQ = \psi'.$$

Then, for the motions of P, Q, R, we have

$$P\left(\frac{d^2x}{dt^2} - g\cos\delta_1\right) + T(\cos\phi + \cos\phi') = 0 \quad \text{..........(1)},$$

$$Q\left(\frac{d^2y}{dt^2} - g\cos\delta_2\right) + T(\cos\chi + \cos\chi') = 0 \quad \text{..........(2)},$$

$$R\left(\frac{d^2z}{dt^2} - g\cos\delta_3\right) + T(\cos\psi + \cos\psi') = 0 \quad \text{..........(3)}.$$

And eliminating T,

$$P(\cos\psi + \cos\psi')\left(\frac{d^2x}{dt^2} - g\cos\delta_1\right) - R(\cos\phi + \cos\phi')\left(\frac{d^2z}{dt^2} - g\cos\delta_3\right) = 0 \quad \text{........(4)},$$

$$Q(\cos\psi + \cos\psi')\left(\frac{d^2y}{dt^2} - g\cos\delta_2\right) - R(\cos\chi + \cos\chi')\left(\frac{d^2z}{dt^2} - g\cos\delta_3\right) = 0 \quad \text{........(5)}.$$

These are the general equations of motion, but their complete solution cannot be effected except for small oscillations.

The geometrical relations give

$$u\cos\phi + y\cos\alpha = x, \quad v\cos\chi + z\cos\beta = y, \quad w\cos\psi + x\cos\gamma = z,$$
$$w\cos\phi' + z\cos\gamma = x, \quad u\cos\chi' + x\cos\alpha = y, \quad v\cos\psi' + y\cos\beta = z.$$

Let a, b, c be the values of x, y, z, when there is equilibrium; a', b', c' the values of u, v, w; and $\lambda, \lambda', \mu, \mu', \nu, \nu'$ those of $\cos\phi$, $\cos\phi'$, $\cos\chi$, $\cos\chi'$, $\cos\psi$, $\cos\psi'$.

Assume $$x = a + x', \quad y = b + y', \quad z = c + z' \quad \ldots\ldots\ldots\ldots\ldots (6),$$
$$u = a' + u', \quad v = b' + v', \quad w = c' + w' \quad \ldots\ldots\ldots\ldots\ldots (7),$$
where x', y', z', u', v', w' are small quantities.

Now $u^2 = x^2 + y^2 - 2xy\cos\alpha$, $v^2 = y^2 + z^2 - 2yz\cos\beta$, $w^2 = z^2 + x^2 - 2zx\cos\gamma$; or, neglecting small quantities of orders higher than the first,
$$a'u' = (a - b\cos\alpha)x' + (b - a\cos\alpha)y', \quad b'v' = (b - c\cos\beta)y' + (c - b\cos\beta)z',$$
$$c'w' = (c - a\cos\gamma)z' + (a - c\cos\gamma)x'.$$
But $a - b\cos\alpha = a'$, &c.,

therefore $$u' = \lambda x' + \mu' y', \quad v' = \mu y' + \nu' z', \quad w' = \nu z' + \lambda' x' \quad \ldots\ldots\ldots\ldots (8).$$
Hence, from (7),
$$u = a' + \lambda x' + \mu' y', \quad v = b' + \mu y' + \nu' z', \quad w = c' + \lambda' x' + \nu z'.$$
Since $u' + v' + w' = 0$, equations (8) give
$$(\lambda + \lambda')x' + (\mu + \mu')y' + (\nu + \nu')z' = 0;$$
therefore $$(\lambda + \lambda')\frac{d^2x'}{dt^2} + (\mu + \mu')\frac{d^2y'}{dt^2} + (\nu + \nu')\frac{d^2z'}{dt^2} = 0.$$

Also, from (6), $$\frac{d^2x}{dt^2} = \frac{d^2x'}{dt^2}, \quad \frac{d^2y}{dt^2} = \frac{d^2y'}{dt^2}, \quad \frac{d^2z}{dt^2} = \frac{d^2z'}{dt^2}.$$

Substituting these in (4) and (5), neglecting small quantities of orders higher than the first, and putting
$$A = \frac{\lambda}{a'}(\nu + \nu') + \frac{1}{c'}(\lambda'\nu' - \cos\gamma) - \left(\frac{1}{b'} + \frac{1}{c'}\right)(1 + \nu\nu')\frac{\lambda + \lambda'}{\nu + \nu'},$$
$$B = \left\{\frac{\nu'}{b'}(\lambda + \lambda') + \frac{1}{c'}(\lambda\nu - \cos\gamma)\right\}\frac{\lambda + \lambda'}{\nu + \nu'} - \left(\frac{1}{a'} + \frac{1}{c'}\right)(1 + \lambda\lambda'),$$
$$C = \frac{\mu'}{a'}(\nu + \nu') + \frac{1}{b'}(\mu\nu - \cos\beta) - \left(\frac{1}{b'} + \frac{1}{c'}\right)(1 + \nu\nu')\frac{\mu + \mu'}{\nu + \nu'},$$
$$D = \left\{\frac{\nu'}{b'}(\lambda + \lambda') + \frac{1}{c'}(\lambda\nu - \cos\gamma)\right\}\frac{\mu + \mu'}{\nu + \nu'} - \frac{\mu}{b'}(\lambda + \lambda') - \frac{1}{a'}(\lambda'\mu' - \cos\alpha),$$
$$A' = \frac{\lambda}{a'}(\nu + \nu') + \frac{1}{c'}(\lambda\nu' - \cos\gamma) - \left(\frac{1}{b'} + \frac{1}{c'}\right)(1 + \nu\nu')\frac{\lambda + \lambda'}{\nu + \nu'},$$
$$B' = \left\{\frac{\nu}{c'}(\mu + \mu') + \frac{1}{b'}(\mu'\nu' - \cos\beta)\right\}\frac{\lambda + \lambda'}{\nu + \nu'} - \frac{\lambda'}{c'}(\mu + \mu') - \frac{1}{a'}(\lambda\mu - \cos\alpha),$$
$$C' = \frac{\mu'}{a'}(\nu + \nu') + \frac{1}{b'}(\mu\nu' - \cos\gamma) - \left(\frac{1}{b'} + \frac{1}{c'}\right)(1 + \nu\nu')\frac{\mu + \mu'}{\nu + \nu'},$$
$$D = \left\{\frac{\nu}{c'}(\mu + \mu') + \frac{1}{b'}(\mu'\nu' - \cos\beta)\right\}\frac{\mu + \mu'}{\nu + \nu'} - \left(\frac{1}{a'} + \frac{1}{b'}\right)(1 + \mu\mu'),$$
we find, after reduction,
$$\frac{P(\nu + \nu')^2 + Q(\lambda + \lambda')^2}{\nu + \nu'} \cdot \frac{d^2x'}{dt^2} + \frac{R(\lambda + \lambda')(\mu + \mu')}{\nu + \nu'} \cdot \frac{d^2y'}{dt^2}$$
$$= g(PA\cos\delta_1 + RB\cos\delta_3) + g(PC\cos\delta_1 + RD\cos\delta_3),$$

$$\frac{\mathrm{R}(\lambda+\lambda')(\mu+\mu')}{\nu+\nu'}\cdot\frac{d^2x'}{dt^2}+\frac{\mathrm{Q}(\nu+\nu')^2+\mathrm{R}(\mu+\mu')^2}{\nu+\nu'}\cdot\frac{d^2y'}{dt^2}$$
$$= g(\mathrm{QA}'\cos\delta_2+\mathrm{RB}'\cos\delta_3)+g(\mathrm{QC}'\cos\delta_2+\mathrm{RD}'\cos\delta_3).$$

Again, putting

$$h = \mathrm{QR}(\lambda+\lambda')^2+\mathrm{RP}(\mu+\mu')^2+\mathrm{PQ}(\nu+\nu')^2,$$
$$k = \left\{\mathrm{Q}(\nu+\nu')^2+\mathrm{R}(\mu+\mu')^2\right\}(\mathrm{PA}\cos\delta_1+\mathrm{RB}\cos\delta_3) - \mathrm{R}(\lambda+\lambda')(\mu+\mu')(\mathrm{QA}'\cos\delta_2+\mathrm{RB}'\cos\delta_3),$$
$$l = \left\{\mathrm{R}(\mu+\mu')^2+\mathrm{Q}(\nu+\nu')^2\right\}(\mathrm{PC}\cos\delta_1+\mathrm{RD}\cos\delta_3) - \mathrm{R}(\lambda+\lambda')(\mu+\mu')(\mathrm{QC}'\cos\delta_2+\mathrm{RD}'\cos\delta_3),$$
$$k' = \left\{\mathrm{R}(\lambda+\lambda')^2+\mathrm{P}(\nu+\nu')^2\right\}(\mathrm{QA}'\cos\delta_2+\mathrm{RB}'\cos\delta_3) - \mathrm{R}(\lambda+\lambda')(\mu+\mu')(\mathrm{PA}\cos\delta_1+\mathrm{RB}\cos\delta_3),$$
$$l' = \left\{\mathrm{P}(\nu+\nu')^2+\mathrm{R}(\lambda+\lambda')^2\right\}(\mathrm{QC}'\cos\delta_2+\mathrm{RD}'\cos\delta_3) - \mathrm{R}(\lambda+\lambda')(\mu+\mu')(\mathrm{PC}\cos\delta_1+\mathrm{RD}\cos\delta_3),$$

we finally get
$$h(\nu+\nu)\frac{d^2x'}{dt^2}-g(kx'+ly')=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(9),$$
$$h(\nu+\nu')\frac{d^2y'}{dt^2}-g(k'x'+l'y')=0 \quad\ldots\ldots\ldots\ldots\ldots(10).$$

Hence, for small oscillations, the motion of the system is given by the simultaneous equations (9), (10), and will consist of two independent vibrations, of which the periods are

$$\pi\left\{\frac{h(\nu+\nu')}{g(k+k'\eta')}\right\}^{\frac{1}{2}},\quad \pi\left\{\frac{h(\nu+\nu')}{g(k+k'\eta'')}\right\}^{\frac{1}{2}};$$

where η', η'' are the roots of the equation $k'\eta^2+(k-l')\eta = l$.

2642. (Proposed by C. W. Merrifield, F.R.S.)—If three surfaces of the second degree meet in four points, at each of which their three curves of intersection have a common tangent, these four points lie in one plane.

Solution by the Proposer.

Write the equation of the quadric in its general form

$$\mathrm{F} = ax^2+by^2+cz^2+d^2+2lyz+2mzx+2nxy+2px+2qy+2rz=0,$$

and use suffixes to distinguish the three surfaces F_1, F_2, F_3.

Taking our origin at one of the points, and making the axes of x and y pass through two others $(2h, 0, 0)$ and $(0, 2k, 0)$, we get $d=0$ and $p=-ah$, $q=-bk$, where h and k are independent of the suffix. Take, moreover, the common tangent at the origin for the axis of z, and we have r_1, r_2, r_3 each $=0$.

The tangent planes at the origin may therefore be written as

$$a_1hx+b_1ky=0,\quad a_2hx+b_2ky=0,\quad a_3hx+b_3ky=0;$$

and as these are to have a common intersection, we get

$$\frac{a_1-a_2}{a_3}=\frac{b_1-b_2}{b_3}.$$

Treating in a similar manner the points $(2h, 0, 0)$ and $(0, 2k, 0)$, for which the tangent planes are

$$(x-2h)\,ah\ +y\,(2nh-bk)+z\,.\,2mh=0$$

and
$$x\,(2nk-ah)+(y-2k)\,bk\ +z\,.\,2lk\ =0,$$

we get the further relations

$$\frac{a_1-a_2}{a_3}=\frac{m_1-m_2}{m_3}=\frac{2\,(n_1-n_2)\,h-(b_1-b_2)\,k}{2n_3h-b_3k},$$
$$\frac{b_1-b_2}{b_3}=\frac{l_1-l_2}{l_3}=\frac{2\,(n_1-n_2)\,k-(a_1-a_2)\,h}{2n_3k-a_3h},$$

and since $\dfrac{a_1-a_2}{a_3}=\dfrac{b_1-b_2}{b_3}$, these are all equal, and so is also $\dfrac{n_1-n_2}{n_3}$. We thus get the following reductions in the general equation:—

$$d=0,\quad r=0,\quad \frac{p_1}{a_1}=\frac{p_2}{a_2}=\frac{p_3}{a_3}=-h,\quad \frac{q_1}{a_1}=\frac{q_2}{a_2}=\frac{q_3}{a_3}=-k,$$
$$\frac{a_1-a_2}{a_3}=\frac{b_1-b_2}{b_3}=\frac{l_1-l_2}{l_3}=\frac{m_1-m_2}{m_3}=\frac{n_1-n_2}{n_3}(=g);$$

and we may therefore write the three quadrics as

$$F_1=a_1x^2+b_1y^2+c_1z^2+2l_1yz+2m_1zx+2n_1xy-2a_1hx-2b_1ky=0,$$
$$F_2=a_2x^2+\ldots\ldots$$
$$F_3=\frac{a_1-a_2}{g}x^2+\frac{b_1-b_2}{g}y^2+c_3z^2+2\frac{l_1-l_2}{g}yz+\ldots\ldots\ldots=0,$$

the only outlying term being c_3.

But
$$F_3-\frac{F_1-F_2}{g}=0\ \text{ gives }\ \frac{gc_3-c_1+c_2}{g}z^2=0,$$

or there is a quadric through the intersection which degenerates into the double plane $z=0$. This proves the theorem, which is easily reciprocated. The investigation, moreover, shows that, if there be three such points, there will also be a fourth.

2771. (From WOLSTENHOLME'S *Book of Mathematical Problems.*)—Prove that (1) if a straight line of length a be divided at random in two points, the mean value of the sum of the squares on the three parts is $\frac{1}{2}a^2$; (2) if the line be divided at random into two parts, and the longer part again divided at random into two parts, the mean value of the sum of the squares

on the three parts is $\frac{35}{72}a^2$; and (3) if it is an even chance that n times the sum of the squares on the parts in (1) is less than the square on the whole line, then $n = \dfrac{12\pi}{4\pi + 3\sqrt{3}}$.

I. *Solution by* STEPHEN WATSON.

(1.) Let x, y, and $a-x-y$ be the parts. Then the total of cases is a^2, and doubling because of the interchange of x and y, the average is

$$\frac{2}{a^2}\int_0^a dx \int_0^{a-x} \left\{x^2+y^2+(a-x-y)^2\right\} dy = \tfrac{1}{2}a^2.$$

(2.) Here the number of cases is

$$2\int_0^{\frac{1}{2}a} (a-x)\, dx = \tfrac{3}{4}a^2,$$

and the average is

$$2\int_0^{\frac{1}{2}a} dx \int_0^{a-x} \left\{x^2+y^2+(a-x-y)^2\right\} dy \div \tfrac{3}{4}a^2 = \tfrac{35}{72}a^2.$$

(3.) In this case, let x, $\frac{1}{2}(a-x)-y$, $\frac{1}{2}(a-x)$ be the parts; then

$$x^2 + \tfrac{1}{2}(a-x)^2 + 2y^2 < \frac{a^2}{n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

therefore $\qquad y < \pm\tfrac{1}{2}(ma^2-3z^2)^{\frac{1}{2}}$,

where $m = \dfrac{2}{n} - \dfrac{2}{3}$ and $z = x - \frac{1}{3}a$. Hence the number of positions y can take for each value of x is $(ma^2-3z^2)^{\frac{1}{2}}$, and therefore the chance p of (1) being fulfilled is, since $dx = dz$,

$$p = \frac{2}{a^2}\int_{-(\frac{1}{3}m)^{\frac{1}{2}}a}^{(\frac{1}{3}m)^{\frac{1}{2}}a} (ma^2-3z^2)^{\frac{1}{2}}\, dz = \frac{m\pi}{\sqrt{3}} = \left(\frac{2}{n}-\frac{2}{3}\right)\frac{\pi}{\sqrt{3}}.$$

therefore $\qquad n = \dfrac{6\pi}{2\pi + 3p\sqrt{3}}$, (when $p = \frac{1}{2}$) $= \dfrac{12\pi}{4\pi + 3\sqrt{3}}$.

II. *Solution by the* REV. J. WOLSTENHOLME, M.A.

(1.) The answer to this part of the question is

$$\int_0^a\int_x^a \left\{x^2+(y-x)^2+(a-y)^2\right\} dx\,dy \div \int_0^a\int_x^a dx\,dy$$

$$= 2a^2\int_0^1 \left\{(1+2x^2)(1-x)-(1+x)^2(1-x^2)+\tfrac{2}{3}(1-x^3)\right\} dx$$

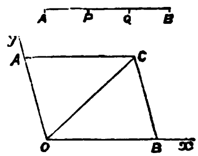
$$= \tfrac{2}{3}a^2\int_0^1 (2-6x+9x^2-5x^3) = 2a^2(2-3+3-\tfrac{5}{4}) = \tfrac{1}{2}a^2.$$

(2.) The answer to this part of the question is

$$\int_0^{\frac{a}{2}}\int_0^{a-x}\left\{x^2+y^2+(a-x-y)^2\right\}dx\,dy \div \int_0^{\frac{a}{2}}\int_0^{a-x}dx\,dy$$

$$=\frac{8a^2}{3}\int_0^{\frac{1}{2}}\left\{x^2+\tfrac{2}{3}(1-x)^2\right\}(1-x)\,dx = \frac{8a^2}{9}\int_0^{\frac{1}{2}}(2-6x+9x^2-5x^3)\,dx$$

$$=\frac{8a^2}{9}\left(1-\frac{6}{8}+\frac{9}{24}-\frac{5}{64}\right)=\frac{a^2}{72}(64-48+24-5)=\frac{35}{72}a^2.$$



(3.) If AB be the rod; P, Q the points of division, P nearer A; AP $= x$, AQ $= Y$; then let us find the chance that $\quad x^2+(y-x)^2+(a-y)^2 < \dfrac{a^2}{n}$,

where $x > 0 < a$, $y > x < a$.

Take axes Ox, Oy inclined at 120°; measure OA = OB = a, and complete the rhombus OACB; then to every possible division of the rod corresponds a point in the triangle OAC, and to every favourable division a point within the circle $\qquad x^2+y^2-xy-ay+\dfrac{a^2}{2}=\dfrac{a^2}{2n}$;

or, transferring to the centre $\mathrm{X}^2-\mathrm{XY}+\mathrm{Y}^2=\dfrac{a^2}{2n}-\dfrac{a^2}{6}$.

When $n=2$, this is the inscribed circle; and when $n>2$, the circle lies altogether within the triangle, and

its area : area of inscribed circle $=\dfrac{1}{n}-\dfrac{1}{3}:\dfrac{1}{6}=2(3-n):n$,

and area of inscribed circle : area of triangle $=\pi : 3\sqrt{3}$;

therefore chance we are seeking is $\dfrac{2\pi(3-n)}{3n\sqrt{3}}=\dfrac{1}{2}$, if $n=\dfrac{12\pi}{4\pi+3\sqrt{3}}$;

and since this is greater than 2, it is the value required. If $n<2$, the chance that n times sum of squares is less than the square on whole is

$$\frac{3-n}{3n\sqrt{3}}(3\sin 2\alpha+2\pi-6\alpha),\quad\text{where }\sin\alpha=\left(\frac{6-3n}{6-2n}\right)^{\frac{1}{2}}.$$

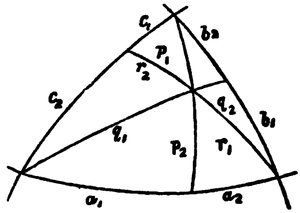
The chance that 3 times squares on parts > 2 square on whole is $\dfrac{\pi+6}{6\sqrt{3}}$,

" 7 " " > 5 square on whole is $\dfrac{1}{3}+\dfrac{2\pi}{9\sqrt{3}}$.

2455. (Proposed by the Rev. R. Townsend, F.R.S.)—If a, b, c be the three sides of a spherical triangle, and k the radius of its polar circle, prove the formula

$$\tan^2 k=\frac{(\cos b\cos c\sec a-1)(\cos c\cos a\sec b-1)(\cos a\cos b\sec c-1)}{4\sin s\sin(s-a)\sin(s-b)\sin(s-c)}.$$

Solution by the PROPOSER.



If p, q, r be the three perpendiculars of the triangle; a_1 and a_2, b_1 and b_2, c_1 and c_2 the three pairs of segments into which they divide the three sides; A_1 and A_2, B_1 and B_2, C_1 and C_2 the three pairs into which they divide the three angles; p_1 and p_2, q_1 and q_2, r_1 and r_2 the three pairs into which they divide each other; and k as above; then, since

$$\tan p_1 \tan p_2 = \tan q_1 \tan q_2 = \tan r_1 \tan r_2 = -\tan^2 k,$$

and since, by Napier's rule,

$$\tan p \tan p_1 = \tan c_1 \tan c_2 = \tan b \tan c \cos A$$

$$\tan p_2 = \sin a_1 \tan B_1 = \tan p \cot B \cot C \sec a,$$

therefore, at once, $\tan^2 k = -\dfrac{\sin b \sin c}{\sin B \sin C} \cdot \dfrac{\cos A \cos B \cos C}{\cos a \cos b \cos c}$,

which, substituting for the functions of the angles their familiar values in terms of the sides, gives immediately the above.

A process exactly similar leads to the corresponding formula for a plane triangle; viz.,

$$k^2 = -\frac{\sin b \sin c}{\sin B \sin C} \cos A \cos B \cos C = -\frac{(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}{32s(s-a)(s-b)(s-c)}$$

[See Townsend's *Modern Geometry*, Vol. I., art. 168.]

2877. (Proposed by Professor SYLVESTER.)—If

$$u = x+y+z+t, \qquad v = fx+gy+hz+kt,$$

$$U = yzt+ztx+txy+xyz, \qquad V = fyzt+gztx+htxy+kxyz;$$

prove that the resultant of u, v, U, V is

$$(f-g)(f-h)(f-k)(g-h)(g-k)(h-k)(f+g-h-k)^2(f+h-g-k)^2 \times (f+k-g-h)^2.$$

Solution by the PROPOSER.

Suppose $x+y=0$, $z=0$, $t=0$; then all four equations are satisfied provided $f=g=0$. Hence $(f-g)(f-h)(g-h)(f-k)(g-k)(h-k)$ is a factor of the resultant.

Again, suppose $x=y=-z=-t$; all four equations are satisfied provided $f+g=h+k$. Hence $(f+g-h-k)(f+h-g-k)(g+h-f-k)$ is also a factor of the same.

But furthermore, if $f+g-h-k=0$, then at the point of intersection $x=1$, $y=1$, $z=-1$, $t=-1$, it may easily be verified that the planes u, U

respectively touch the cubic surfaces v, V. Either of these facts warrants us in concluding that some power of $(f+g-h-k)$ ***higher than the first*** must enter into the resultant. Hence

$$(f-g)(f-h)(f-k)(g-h)(g-k)(h-k)$$
$$\times\left\{(f+g-h-k)(f+h-g-k)(g+h-f-k)\right\}^2$$

is contained in the resultant. But the order of the resultant in these letters is evidently 3^2+3, *i.e.* 12. Hence the above quantity is the complete resultant.

I obtained the theorem originally by ordinary Algebra, I forget how, probably by a direct application of the dialytic process; but the preceding method is more instructive, as embodying in a simple instance the transcendental law familiar to eliminationists, that ***any singularity of relation between loci intersecting in virtue of a condition*** implies the appearance of a power, higher than the first, of the characteristic of that condition in the complete resultant. The theorem was wanted in an enquiry connected with the subject of rectifiable compound logarithmic waves. A vast extension of it will be given in the answer to a subsequent question.

NOTE ON THE LATE JUDGE HARGREAVE'S SOLUTION OF THE QUINTIC.

By the Rev. T. P. KIRKMAN, M.A., F.R.S.

One of the final equations at page 94 of the Judge's posthumous treatise is

$$y_1^{\frac{1}{5}}+y_2^{\frac{1}{5}} = A_1(z_1-b_1)+A_2(z_2-b_1)+A_3(z_3-b_1)+A_4(z_4-b_1)+A_5(z_5-b_1)$$
$$= \Sigma(At).$$

If so, it is possible to find $\alpha_1+\beta_1 = A_1$, $\alpha_2+\beta_2 = A_2$, &c., such that

$$y_1^{\frac{1}{5}}+y_2^{\frac{1}{5}} = \left\{(\Sigma\alpha t)^5\right\}^{\frac{1}{5}}+\left\{(\Sigma\beta t)^5\right\}^{\frac{1}{5}} = (P^5)^{\frac{1}{5}}+(Q^5)^{\frac{1}{5}},$$

where $P^5 = y_1$, and $Q^5 = y_2$.

As $y_1+y_2 = 2\,(51)$, p. 85, is a rational and symmetrical function of the z's, P^5+Q^5 is one also; that is, P^5 is a two-valued function, whose values are P^5 and Q^5. It follows that P is also a two-valued function of the z's, whose two values are P and Q; for if P had more than these two values, PPPPP + QQQQQ could not be a rational and symmetrical function. Hence $P+Q = \Sigma(At)$ is a symmetrical function of the z's. That is,

$$y_1^{\frac{1}{5}}+y_2^{\frac{1}{5}} = \Sigma(At) = A\Sigma t = A\Sigma(z-b_1) = 0,$$

and $\qquad y_1 = -y_2, \quad \text{or} \quad y_1+y_2 = 2\,(51) = 0.$

This destroys the form of the conditioned quintic in z, at p. 82; for $(51)=0$ gives, as (21) and (31) are each $=0$, (p. 78), (p. 7),

$$(51) = 4b_1^5-5b_1b_4+b_5 = 0,$$

or b_5 is expressible in b_1 and b_4. Thus it seems that what HARGREAVE has been solving is not the general quintic.

2870. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—Given three points A, B, C, and a conic; two points P, Q are taken on the conic such that the pencil $A\{BPQC\}$ is harmonic; prove that the envelope of PQ is a conic touching AB, AC at points on the polar of A with respect to the given conic.

Solution by ARCHER STANLEY.

Since the intersections x and x' of every connector PQ and the given lines AB, AC are harmonic conjugates with respect to P and Q, the polar of one must pass through the other; in other words, x and x' are conjugate points relative to the conic. Conversely, every connector of a pair of conjugate points x, x' on AB and AC, being harmonically divided by its intersections P, Q with the conic, must be a tangent of the required envelope. But to a point x on AB there is but one conjugate x' on AC, and *vice versâ*; hence x and x' are corresponding points of two homographic rows, and their connector xx' envelopes a conic which touches AB, AC at the points which correspond to A, that is to say, at the points B, C, where AB and AC are intersected by the polar of A. This envelope has obviously, in common with the given conic, the four tangents at the points where AB and AC cut that conic.

The following are amongst the most interesting special cases:

1. If ABC be a self-conjugate triangle, the envelope degenerates to the *point-pair* (B, C).

2. The envelope also degenerates to a point-pair (A, A′), if A be on the given conic, and A′ be the intersection of the tangents at the points where AB and AC cut the conic again.

3. If AB and AC touch the conic in B and C, the envelope will likewise touch it in these points.

4. If A be at the centre of the given conic, the envelope will be a concentric conic having AB, AC for its asymptotes. If AB, AC be themselves asymptotes of the given conic, then the envelope will be a concentric, similar, and similarly placed conic.

Many well known theorems in conics immediately follow from the above special cases, wherein, it should be observed, AB and AC may be imaginary. For instance, if these lines pass through the circular points at infinity, then the envelope under consideration will become that of chords of the given conic which subtend a right angle at A, and this envelope will have A for its focus. If A be itself a focus of the given conic, then the envelope (by 3) will have in common with it this focus and the corresponding directrix. If A be on the conic, however, then (by 2) the chords which subtend a right angle at A will all pass through a point A′ on the normal at A. (Salmon's *Conics*, Art. 181, Ex. 2.)

2828. (Proposed by S. TEBAY, B.A.)—A straight rod is divided into n parts in arithmetical progression, and equal particles are fixed at the points of division. If the system be made to vibrate about one extremity, determine the average length of the pendulum, neglecting the weight of the rod.

Solution by the PROPOSER.

Let a be the length of the rod, L the length of the pendulum for any adjustment of the particles; $u_1, u_2, \ldots . u_{n-1}$ their distances from the point of suspension. Then

$$\mathrm{L} = \frac{u_1^2 + u_2^2 + \ldots . + u_{n-1}^2}{u_1 + u_2 + \ldots . + u_{n-1}} = \frac{\Sigma(u_n^2)}{\Sigma(u_n)}.$$

Let $u_1 = x$, $x_2 - u_1 = x + y$, where y is the common difference.

Therefore
$$u_n = \tfrac{1}{2}n\left\{2x + (n-1)y\right\};$$

$$\Sigma(u_n^2) = \tfrac{1}{24}n(n-1)(2n-1)(2x-y)^2 + \tfrac{1}{8}n^2(n-1)^2(2x-y)y$$
$$+ \tfrac{1}{120}n(n-1)(2n-1)(3n^2-3n-1)y^2,$$

$$\Sigma(u_n) = \tfrac{1}{4}n(n-1)(2x-y) + \tfrac{1}{12}n(n-1)(2n-1)y.$$

But $\quad \tfrac{1}{2}n\left\{2x + (n-1)y\right\} = a; \quad$ therefore $\; y = \dfrac{2(a-nx)}{n(n-1)};$

therefore
$$\Sigma(u_n^2) = \frac{1}{30n(n-1)}\left\{\frac{n^4-1}{n-1}n^2x^2 + an(n-2)(n+1)(3n+1)x\right.$$
$$\left. + 2a^2(n-2)(3n^2-6n+1)\right\},$$

$$\Sigma(u_n) = \tfrac{1}{6}n(n+1)x + \tfrac{1}{3}a(n-2).$$

Now the number of pendulums is proportional to $\dfrac{a}{n}$, this being the greatest value of x. Therefore the average length of the pendulum is

$$\frac{n}{a}\int_0^{\frac{a}{n}} \frac{\Sigma(u_n^2)}{\Sigma(u_n)} = \frac{a}{10n(n^2-1)}\left\{3n^3 + 5n^2 - 17n + 5\right.$$
$$\left. + 4\frac{(n-1)^2(n-2)(2n-1)}{n+1}\log\frac{3(n-1)}{2(n-2)}\right\}.$$

If n be increased indefinitely, this becomes

$$\frac{a}{5}\left(\frac{3}{2} + 4\log\frac{3}{2}\right) = \cdot 6243\,a.$$

If the distances of the particles from the point of suspension be in arithmetical progression, and one particle be attached to the lower extremity of the rod, the average length of the pendulum is

$$\frac{a}{3(n-1)}\left\{\tfrac{1}{2}(2n-7) + 2(n+1)\log 2\right\}.$$

2533. (Proposed by Professor HIRST.) — Find the envelope of a line upon which two given conics intercept segments which have a common middle point; and find also the locus of this middle point.

I. *Solution by the* Rev. J. WOLSTENHOLME, M.A.

If the two conics be $lx^2+my^2+nz^2=0$, $l'x^2+m'y^2+n'z^2=0$, and $ux+vy+wz=0$ be a straight line divided as required, we must have

$$(mn'-m'n)\frac{u}{v-w}+(nl'-n'l)\frac{v}{w-u}+(lm'-l'm)\frac{w}{u-v}=0,$$

the tangential equation of the envelope. For the locus of the middle point, we have "The envelope of a chord of a conic bisected by a given straight line is a parabola of which that straight line is a tangent." Hence the points in which a given straight line will meet the locus are the points in which it will be met by the two remaining common tangents to two parabolas; two points: or the locus is a conic.

II. *Solution by the* PROPOSER.

Let (C) and (C′) be the two given conics, L one of the lines whose envelope is required, and l the common middle point of the two segments which (C) and (C′) determine upon L. It is obvious that the intersections of (C) and (C′) with L determine an involution of which l is a double point, the other being at infinity; and it is well known that every conic passing through the intersections of (C) and (C′) will cut L in a pair of conjugate points of this involution, that is to say, will intercept on L a segment which has the same point l for its middle point. One of the conics of this pencil, in fact, will touch L at l, another will have L for its asymptote, and consequently have its centre on L. Now the locus of the centres of conics of the pencil being a conic (Σ), there will be another conic which has likewise its centre on L, and that centre must obviously be the common middle point l.

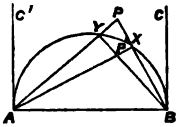
Conversely, every point on (Σ) is a point l on some line L. For, besides being the centre of a conic of the pencil, it is the middle point of one chord of (C). Two conics of the pencil, therefore, intercept segments on this chord which have a common middle point l; hence all do so, and the chord is a position of L. The required locus of l, therefore, is the conic of centres Σ.

Moreover, the above investigation has shown that the required envelope is that of the asymptotes of the conics of the pencil $\{(C), (C')\}$. Hence its class can be easily shown to be 3. For through an arbitrary point x at infinity only one conic of the pencil passes, and therefore only one proper asymptote. The pencil however contains two parabolas, so that the line at infinity must be counted twice as an asymptote through x.

The required envelope being of the third class, and having a double tangent at infinity must, of course, be of the fourth order.

2835. (Proposed by T. COTTERILL, M.A.)—In a plane, if A, B are two points, and a point P describe a curve of the nth order, show that P′, the intersection of the perpendiculars in the triangle PAB, will describe a curve of the $2n$th order with three multiple points of the order n. Explain why the curve corresponding to a circle through the points A and B includes its reflexion to the line AB.

Solution by the PROPOSER.



(1.) Let AC, BC be the perpendiculars to the line AB at the points A, B. Then, if P be a point in the plane, not on the lines, and PA, PB be joined, the perpendiculars AX, BY on PB, PA will intersect in a point P′; the pair will conjugately correspond, since each point is derived from the other by the same construction. The pair lie on a line through C, which is made up of such pairs of points; amongst the lines is the line at infinity.

Let A, B, C be called principal points. Then, if P lie on the line AC, P′ coincides with A. Similarly, to a non-principal point on BC the point B always corresponds. If P is a non-principal point on AB, the point C corresponds. Thus to every non-principal point of the plane one and but one point corresponds, whilst to each of the principal points the points of a line or the line itself corresponds. To a curve, therefore, of the nth order, which does not pass through a principal point, a proper curve of the $2n$th order will correspond, having multiple points of the order n at the points A, B, C. The nature of the correspondence of such curves is more clearly seen by observing (2) that since the angles at X and Y are right angles, the points X and Y lie on the circle described on AB as a diameter, and that P, P′ are harmonic conjugates to the circle. But the line PP′ always passes through C, the pole of the diameter AB; so that we have a particular case of quadric inversion, the general principles of which have been fully explained by Prof. HIRST in his Paper on Quadric Inversion contained in the *Proceedings of the Royal Society* for March, 1865. Corresponding points are therefore inverse points. One peculiarity of this inversion is that a right hyperbola through A, B is its own inverse; and since the pole of PP′ to such a conic passing through PP′ is on XY, it follows that tangents to a curve at P and its inverse at P′ meet on XY.

(3.) The algebraical equations of the correspondence are very simple. Let (u, v, y), (u', v', y') be the perpendiculars from PP′ on the lines AC, BC, and AB, the line PP′ cutting AB in M. Then PM . P′M = AM . MB, so that we have the equations $u' = u$, $v' = v$, and $y'y = uv$. Also

$$y' + \frac{u'v'}{y'} = y + \frac{uv}{y}, \quad y' - \frac{u'v'}{y'} = -\left(y - \frac{uv}{y}\right) \text{ \&c.}$$

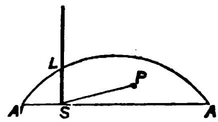
The equation to the circle AXB, the locus of coincident inverse points, is $uv - y^2 = K = 0$. The right hyperbola (axis AB) is $uv + y^2 = H = 0$. If $(u, v, y) = 0$ be the homogeneous equation to a curve, (uy, vy, uv) is its inverse; $(uy, vy, H) = 0$ is its own inverse; the inverse of $(uy, vy, K) = 0$ is its reflexion to the line AB, &c.

In tracing a curve which has a simple inverse, the (xy) system of coordinates is often convenient, in which x is the perpendicular from P on OC, O being the centre of AB, y remaining the same as before. If AB = $2a$, the inverse coordinates are connected by the equations $x' = x$, $y'y = a^2 - x^2$. Thus the inverse of the circle $(y+b)^2 = a^2 - x^2$, which touches the lines AC, BC, is $(by + a^2 - x^2)^2 = (a^2 - x^2)\,y^2$, a quartic curve having cusps at A and B, and a double point at C. If the circle does not cut the line AB, the quartic, by its revolution round OC, will generate a surface in the shape of a bell with its mouth uppermost, containing a bowl which just fits it at the rim.

The inverse of a circle through AB is its reflexion to the line AB, because any point and the reflexion of its inverse to the line AB are on a circle through AB, as is easily seen.

2821. (From WOLSTENHOLME's *Book of Mathematical Problems.*)—Show that the mean value of the distance from one of the foci of all points within a given prolate spheroid is $\frac{1}{4}a(3+e^2)$, $2a$ being the axis and e the eccentricity.

Solution by R. TUCKER, M.A.



Let P be an element of the ellipse at a distance r from the focus (S), and the angle ASP $= \theta$; then, by the revolution of the curve about AS, the element at P generates a small ring equal to $2\pi r \sin\theta\, r\, \Delta\theta\, \Delta r$. The volume (V) generated by ASL $= \frac{1}{3}\pi a^3(1-e^2)(1+e)^2(2-e)$, and the volume (V′) generated by A′SL $= \frac{1}{3}\pi a^3(1-e^2)(1-e)^2(2+e)$;

hence $$\frac{V}{V+V'} = \frac{(2-e)(1+e)^2}{4}, \quad \text{and} \quad \frac{V'}{V+V'} = \frac{(2+e)(1-e)^2}{4}.$$

For the part to the right of the bounding plane through SL, putting $r' = \dfrac{a(1-e^2)}{1-e\cos\theta}$, the average distance of P is

$$\frac{2\pi}{V}\int_0^{\frac{1}{2}\pi}\int_0^{r'} r^3 \sin\theta\, d\theta\, dr = \frac{a}{2}\left(\frac{1+e}{2-e}\right)(3-3e+e^2)\ \ldots\ldots\ldots\ (\alpha).$$

The average distance for the portion to the left is found by changing e into $-e$; thus it equals $$\frac{a}{2}\left(\frac{1-e}{2+e}\right)(3+3e+e^2)\ \ldots\ldots\ldots\ (\beta).$$

The average distance required must be determined proportionally to the volumes; thus it equals

$$\frac{\alpha V + \beta V'}{V+V'} = \frac{a}{8}\left\{\frac{(1+e)^3}{e}\left\{1-(1-e)^3\right\} + \frac{(1-e)^3}{e}\left\{(1+e)^3-1\right\}\right\} = \frac{(3+e^2)a}{4}.$$

2815. (Proposed by A. MARTIN.)—A cask contains a gallons of wine. Through a hole in the top water or wine can be let in at the rate of b gallons per minute; and through a pipe in the bottom, when open, the mixture can escape at the same rate. Suppose the discharge pipe is opened at the same instant that water is let in at the top, and t minutes afterwards the water is shut off and wine let in. Required the quantity of water in the cask at the end of t_1 minutes from the opening of the discharge pipe, and the length of time elapsed, both before and after wine was let in at the top, when the quantities of the two fluids in the cask were equal, supposing them to mingle perfectly.

Solution by C. R. RIPPIN, M.A.; R. J. NELSON, M.A.; *the* PROPOSER; *and others.*

Suppose both pipes to have been open for any time t; and let t be divided into n equal intervals, and conceive the alteration in the fluid to have been

effected by removing $\frac{bt}{n}$ gallons in each interval and replenishing it with $\frac{bt}{n}$ gallons of water.

Then, if V′ be the number of gals. of wine at the beginning of an interval, and V″ be the number of gals. of wine at the end of the same interval, we have $V'' = V' - \frac{V'}{a} \cdot \frac{bt}{n} = V'\left(1 - \frac{bt}{an}\right) = V'c$ (c being constant);

so that, if V be the quantity of wine at end of time t, then, since there were a gallons at first, we have

$$V = ac^n = a\left(1 - \frac{bt}{an}\right)^n.$$

Hence, making n infinite, we have for the result in the proposed question, that the wine at end of time $t = a\epsilon^{-\frac{bt}{a}}$, and therefore the water $= a\left(1-\epsilon^{-\frac{bt}{a}}\right)$.

Denoting this by a', we have, by the same process, that the water in the cask after a further time $t' = a'\epsilon^{-\frac{bt'}{a}} = a'\epsilon^{-\frac{b}{a}(t_1-t)}$; and the wine and water will be equal when

$$a\epsilon^{-\frac{bt}{a}} = \frac{a}{2} \text{ or } t = \frac{a}{b}\log 2 \ldots\ldots\ldots\ldots\ldots (1),$$

and also when

$$a'\epsilon^{-\frac{bt'}{a}} = \frac{a}{2} \text{ or } t' = \frac{a}{b}\log\frac{2a'}{a} \ldots\ldots\ldots\ldots (2).$$

2816. (Proposed by the EDITOR.)—Find three square numbers in arithmetical progression, such that the square root of each (α) increased or else (β) diminished by unity shall give three rational squares.

I. *Solution by* SAMUEL BILLS.

If we put $a = 2pq + (p^2 - q^2)$, $b = 2pq - (p^2 - q^2)$, and $c = p^2 + q^2$, then will a^2, b^2, c^2 be three square numbers in arithmetical progression.

Now put $\frac{ax}{y^2}, \frac{bx}{y^2}, \frac{cx}{y^2}$ for the roots of the three squares required in the question; then in the part (α) of the question we shall have to find

$\frac{ax}{y^2} + 1$, $\frac{bx}{y^2} + 1$, $\frac{cx}{y^2} + 1$, or $ax + y^2$, $bx + y^2$, $cx + y^2$ all squares.

Assume $ax + y^2 = (r+y)^2 \ldots\ldots (1)$, and $bx + y^2 = (s+y)^2 \ldots\ldots (2)$. From (1) and (2) we readily find

$$x = \frac{rs(r-s)}{as-br}, \text{ and } y = \frac{br^2 - as^2}{2(as-br)}.$$

Substituting these values in $cx+y^2$, it becomes

$$\frac{crs(r-s)}{as-br}+\frac{(br^2-as^2)^2}{4(as-br)^2},$$

which is required to be a square; that is, we must have

$$4crs(r-s)(as-br)+(br^2-as^2)^2 = \text{a square}.$$

The above reduces to

$$b^2r^4-4bcr^3s+(4ac+4bc-2ab)r^2s^2-4acrs^3+a^2s^4 = \text{a square}.$$

Assume the above $=\left\{br^2-2crs+\left(\frac{2ac-2c^2}{b}+2c-a\right)s^2\right\}^2.$

Squaring this expression and comparing it with the above, we have

$$-4acr+a^2s = -4c\left(\frac{2ac-2c^2}{b}+2c-a\right)r+\left(\frac{2ac-2c^2}{b}+2c-a\right)^2s.$$

By reducing this, we find $\qquad r=\frac{a+b-c}{2b}s.$

We give the following numerical examples, remarking that p and q must be so assumed as to make x positive.

1. Take $p=2$ and $q=1$; then $a=7$, $b=1$, $c=5$; whence $r=\frac{3}{2}s$. Take $s=2$, then $r=3$; whence $x=\frac{6}{11}$, and $y=-\frac{19}{22}$. This gives $\frac{264}{361}$, $\frac{1320}{361}$, $\frac{1848}{361}$ for the roots of the required squares.

2. As another example, take $p=7$ and $q=4$; then we get $a=89$, $b=23$, $c=65$; which gives $r=\frac{47}{46}s$. Take $s=46$, then $r=47$, $x=\frac{94}{131}$, $y=-\frac{5979}{262}$; whence we obtain for the three roots, $\frac{1132888}{(5979)^2}$, $\frac{3201640}{(5979)^2}$, $\frac{4383784}{(5979)^2}$.

3. For a third example, take $p=5$, and $q=2$; this gives $a=41$, $b=-1$, $c=29$; or taking b *positive*, as we are at liberty to do, we shall have $a=41$, $b=1$, $c=29$; whence we find $r=\frac{13}{2}s$. If now we take $s=2$, we have $r=13$; whence $x=\frac{286}{69}$, $y=\frac{5}{138}$, $\frac{x}{y^2}=\frac{78936}{25}$. From these values we find the roots of the three squares to be 8157·44, 91565·76, 129455·04.

Another solution may be obtained as follows :—

Put $\qquad a=2pq+(p^2-q^2),\ b=p^2+q^2,\ c=2pq-(p^2-q^2);$

then will a^2, b^2, c^2 be three square numbers in arithmetical progression. Now let $\frac{ax}{y^2}$, $\frac{bx}{y^2}$, $\frac{cx}{y^2}$ be the roots of the three squares required in the question; then we have to determine x and y so that $ax+y^2$, $bx+y^2$, $cx+y^2$ may all be squares.

Assume $\qquad ax+y^2=l^2,\ bx+y^2=m^2,\ cx+y^2=n^2;$

then $\qquad \frac{l^2-y^2}{a}=\frac{m^2-y^2}{b}=\frac{n^2-y^2}{c}=x\ (=rst \text{ suppose});$

therefore $\qquad l^2-y^2=arst,\ m^2-y^2=brst,\ n^2-y^2=crst.$

Assume $l+y=ar,\ l-y=st,\ m+y=bs,\ m-y=rt,\ n+y=ct,\ n-y=rs.$

We must now have $\quad 2y = ar-st = bs-rt = ct-rs.$

From this $$r = \frac{s(t+b)}{t+a} = \frac{t(s+c)}{s+a};$$

whence $$(t+b)s^2+(ab-t^2)s-ct(t+a) = 0,$$

therefore $(t^2-ab)^2+4ct(t+a)(t+b)$ must be a square. This reduces to
$$t^4+4ct^3+2(2ac+2bc-ab)t^2+4abct+a^2b^2;$$

which assume $=(t^2+2ct+2ac+2bc-2c^2-ab)^2$;

then, by reducing this, we find $\quad t = \frac{1}{2}(c-b-a).$

By symmetry, we have $\quad s = \frac{1}{2}(b-c-a), \quad r = \frac{1}{2}(a-b-c).$

Also, $$y = \tfrac{1}{2}(ar-st) = \tfrac{1}{8}(a^2+b^2+c^2-2ab-2ac-2bc);$$

whence $$\frac{x}{y^2} = \frac{8(a+c-b)(a+b-c)(a-b-c)}{(2ab+2ac+2bc-a^2-b^2-c^2)^2}.$$

Therefore the roots of the three required squares will be
$$\frac{8a(a+c-b)(a+b-c)(a-b-c)}{(2ab+2ac+2bc-a^2-b^2-c^2)^2},\ \frac{8b(a+c-b)(a+b-c)(a-b-c)}{(2ab+2ac+2bc-a^2-b^2-c^2)^2},$$
$$\frac{8c(a+c-b)(a+b-c)(a-b-c)}{(2ab+2ac+2bc-a^2-b^2-c^2)^2},$$
where a, b, c must, to satisfy (α), be the roots of three square numbers in arithmetical progression, such that, taking them all positively, the greatest root must *exceed* the sum of the other two.

The results given in the former solution may be readily obtained from this.

If the greatest root be *less* than the sum of the other two, the result would satisfy (β), that is, it would give three square numbers in arithmetical progression, such that, if each of their square roots be taken from unity, the three remainders shall be rational squares.

For example, take $p=3$ and $q=2$; then $a=17$, $b=13$, $c=7$; and we find for the three roots
$$\frac{ax}{y^2} = -\frac{103224}{(355)^2},\quad \frac{bx}{y^2} = -\frac{78936}{(355)^2},\quad \frac{cx}{y^2} = -\frac{42504}{(355)^2}.$$

Other answers may be readily found.

II. *Solution by the* Proposer.

Let x, y, z be the roots of the three required square numbers, then these have to satisfy the conditions
$$x^2+z^2 = 2y^2,\ x+1 = m^2,\ y+1 = n^2,\ z+1 = p^2 \ \ldots\ldots\ldots (1, 2, 3, 4),$$
where m, n, p are rational numbers. From (1), by the aid of (2), (3), (4),

we have $$\frac{x+y}{y+z} = \frac{y-z}{x-y}, \text{ or } \frac{m^2+n^2-2}{n^2+p^2-2} = \frac{(n+p)(n-p)}{(m+n)(m-n)} \ \ldots\ldots\ldots (5).$$

Assume the relations $\quad m+n = r(n+p), \quad m-n = s(n-p);$

then from these we obtain
$$\frac{m}{r-s+2rs} = \frac{n}{r+s} = \frac{p}{2-r+s}\ \left(=\frac{1}{k}\text{ suppose}\right) \ldots\ldots\ldots\ldots (6).$$

Substituting these values in (5), it becomes

$$k^2 = (r-s+rs+1)^2+(rs+1)^2+4\,\frac{r-s}{rs-1}\;\ldots\ldots\ldots\ldots(7);$$

and we have to determine r and s so that the second side of (7) may be a perfect square. In order to effect this, let $rs-1=f$; then, putting for shortness' sake $g=\frac{1}{2}\left(f+2+\dfrac{2}{f}\right)$, (7) becomes

$$k^2 = (r+s)^2+4g\,(r-s+f)\;\;[=(r+s-2g)^2 \text{ suppose}]\;\ldots\ldots\ldots(8).$$

From (8) we find $r=\frac{1}{2}(g-f)$; and thence, by the aid of (6), (2), (3), (4), we obtain the following general expressions for x, y, z, the roots of the required squares:

$$\frac{4\,(rs+r-g)\,(rs-s+g)}{(r+s-2g)^2},\;\;\frac{4g\,(r+s-g)}{(r+s-2g)^2},\;\;\frac{4\,(1-r+g)\,(1+s-g)}{(r+s-2g)^2},$$

where $\quad g=\frac{1}{2}f+1+\dfrac{1}{f},\quad r=\frac{1}{2}(g-f),\quad$ and $\quad s=\dfrac{1+f}{r},$

f being any rational number whatever.

Taking $f=1$, we obtain the numbers given as the results to satisfy the *first* condition (α) when the question was originally proposed, viz.

$$\frac{264}{361},\;\frac{1320}{361},\;\frac{1848}{361}.$$

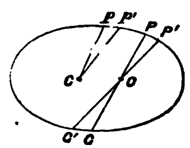
Taking $f=\frac{1}{2}$, we find

$$-x=\frac{42504}{(355)^2},\quad -y=\frac{78936}{(355)^2},\quad -z=\frac{103224}{(355)^2};$$

hence these fractions are the roots of three square numbers in arithmetical progression which satisfy the *second* condition (β) in the question, that is, they are such that, if the roots be taken from unity, the three remainders will be rational squares.

2486. (Proposed by G. O. Hanlon.)—Find a point on an ellipse such that the normal produced to meet the curve again may cut off a maximum area.

I. *Solution by* Professor Whitworth.



Let PG be the required normal meeting the conic again in G. Let P' be a point on the curve adjacent to P, and P'G' the normal chord at P'. Let O be the point of intersection of these two normals. Then ultimately, when P' coincides with P, O is the centre of curvature at P.

Since PG is the normal which divides the area the most unequally, therefore the greater area cut off by PQ is a maximum. Hence ultimately area PP'O = area GG'O. But these areas are ultimately in the ratio of the rectangles OP . OP' and OG . OG'.

Therefore $OP.OP' = OG.OG'$ ultimately. But if $Cp.Cp'$ be semi-diameters parallel to OP and OP′, we have

$$OP.OG : OP'.OG' = Cp^2 : Cp'^2;$$

therefore $OP^2 : OG'^2 = Cp^2 : Cp'^2$, or $OP : OG' = Cp : Cp$.

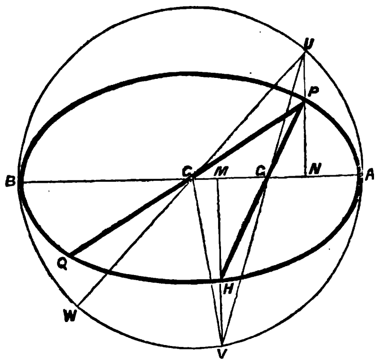
But ultimately $Cp = Cp'$; therefore ultimately $OP = OG'$ or $OP = OG$. Hence PG is the diameter of curvature and the common chord of the ellipse and circle of curvature. Hence PG and the tangent at P make equal angles with the axis, viz., angles of 45°. Therefore the normal required is a normal inclined at an angle of 45° to the axis.

COROLLARY.—The normal which cuts off the smallest closed area from any conic is inclined at an angle of 45° to the axes.

[A solution by Mr. JENKINS has been given on p. 56 of Vol. X. of the *Reprint.*]

II. *Solution by the* EDITOR.

Let C be the centre of the ellipse; PCQ the diameter, and PGH the normal chord through P; NPU and MHV the ordinates at P and H, meeting the circle circumscribing the ellipse in U and V, and UCW a diameter of the circle through U.



Then it may be readily shown that U, G, V are in the same straight line; and by a well known property we have

$$\frac{\text{area } AGP}{\text{area } AGU} = \frac{AGH}{AGV} = \frac{APH}{AUV} = \frac{PHQ}{UVW} = \frac{b}{a},$$

a and b being the semi-axes of the ellipse. Hence, putting S for the area of PHQ, and ϕ for the angle VCW, we have

$$S = \text{area } PHQ = \frac{b}{a}(UVW) = \tfrac{1}{2}ab(\phi + \sin\phi).$$

Now $$\frac{UN}{PN} = \frac{a}{b}, \text{ and } \frac{CN}{GN} = \frac{a^2}{b^2};$$

hence $$\tan UGN = \frac{a}{b}\tan\theta, \text{ and } \tan UCN = \frac{b}{a}\tan\theta;$$

therefore $$\tan\tfrac{1}{2}\phi = \tan CUG = \tan(UGN - UCN) = \frac{a^2 - b^2}{2ab}\sin 2\theta.$$

Hence $S = \tfrac{1}{2}ab(\phi + \sin\phi)$, where $2ab\tan\tfrac{1}{2}\phi = (a^2 - b^2)\sin 2\theta$.

When S is a maximum, we must have

$$\frac{dS}{d\theta} = \frac{dS}{d\phi}\cdot\frac{d\phi}{d\theta} = ab(a^2 - b^2)(1 + \cos\phi)\cos 2\theta = 0;$$

whence we find the angle $PGN = \theta = 45°$.

Hence we see that the normal chord PGH, which makes an angle of 45° with the major axis, divides the ellipse into two segments, one of which (APH) is a *minimum*, and the other (BPH) a *maximum*.

2527. (Proposed by J. WILSON.)—Find the locus of the intersection of two tangents to a circle, the chord of contact of which subtends a right angle at a fixed point; and also of the middle points of a system of chords of a circle, which subtend a right angle at a fixed point; and thence show that the envelope of a system of chords of a conic, which subtend a right angle at the focus, is another conic of coincident focus; and that the locus of the middle points of a system of chords of a conic which subtend a right angle at a fixed point is another conic.

Solution by the Rev. J. WOLSTENHOLME, M.A.

Taking the fixed point for origin, let equation of circle be $(x-c)^2+y^2=a^2$, and let $lx+my=1$ be a chord. The lines joining the points of intersection to the origin have for their equation

$$x^2+y^2-2cx\,(lx+my)+(c^2-a^2)\,(lx+my)^2=0;$$

and if they be at right angles,

$$2-2cl+(c^2-a^2)\,(l^2+m^2)=0.$$

We may then write the equation of two parallel straight lines satisfying the condition in the form

$$2\,(lx+my)^2-2cl\,(lx+my)+(c^2-a^2)\,(l^2+m^2)=0,$$

and the envelope is therefore

$$(2x^2-2cx+c^2-a^2)\,(2y^2+c^2-a^2)=(2xy-cy)^2,\quad\text{or}\quad \frac{(2x-c)^2}{2a^2-c^2}+\frac{2y^2}{a^2-c^2}=1;$$

a conic whose foci are the origin and the centre of the circle. The middle point of the chord being the foot of the perpendicular from the centre, lies on the auxiliary circle of this ellipse, *i.e.* on the circle $(2x-c)^2+4y^2=2a^2-c^2$. The locus of the intersection of tangents at the extremities of such chords is the reciprocal polar of this conic with respect to a circle whose centre is a focus, and is therefore a circle. Of course the envelope of a system of chords of a conic subtending a right angle at any point is a conic with that point as focus; as we get at once by reciprocating the property of the director circle of a conic.

If the equation of any conic be $ax^2+by^2+2hx+2ky+1=0$, and the straight line $lx+my=1$ subtend a right angle at the origin,

$$a+b+2\,(hl+km)+l^2+m^2=0,$$

and the coordinates of the middle point are

$$\frac{X}{kml-hm^2+bl}=\frac{Y}{hlm-kl^2+am}=\frac{1}{bl^2+am^2}=\frac{lX+mY}{bl^2+am^2}$$
$$=\frac{amX-blY}{(am^2+bl^2)\,(kl-hm)},$$

or $\qquad lX+mY=1,\quad l\,(k+lY)=m\,(h+aX);$

therefore $\qquad \dfrac{l}{h+aX}=\dfrac{m}{k+bY}=\dfrac{1}{aX^2+lY^2+hX+kY};$

and the locus of the middle points is apparently a curve of the fourth degree.

2769. (Proposed by A. MARTIN.)—From a point, taken at random within a triangle, perpendiculars are drawn on the sides; find the probability that the triangle formed by joining the feet of these perpendiculars is acute-angled.

Solution by the Rev. J. WOLSTENHOLME, M.A.

In an acute-angled triangle ABC, if O be the centre of the circumscribed circles, and arcs of circles be described on each side touching the radii from O to the ends of the side, the points must lie without each of the arcs. The result does not seem to be very neat, and it becomes more complicated for an obtuse-angled triangle.

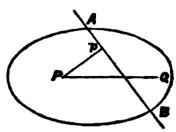
2763. (Proposed by W. S. BURNSIDE, M.A.)—Give a geometrical interpretation for the relation between the invariants of two conics, viz., $\Theta\Theta' = \Delta\Delta'$.

Solution by the Rev. J. WOLSTENHOLME, M.A.

The interpretation is given in Salmon's *Conic Sections*. It is the condition that the 8 points of contact of the common tangents to the two conics may lie on two straight lines; in which case also the 8 tangents to the two conics at their common points will pass through two points, four (two from each conic) through one point and four through the other. An example is any ellipse and a concentric rectangular hyperbola passing through the foci of the ellipse; but the common tangents are unreal. Another example of two circles of radii 1, 7, with distances between centres 10, has the common tangents real, but the common points impossible.

2109. (Proposed by Capt. CLARKE, R.E., F.R.S.)—Two lines are drawn at random across a convex closed curve; determine the chance of their intersecting.

Solution by the PROPOSER.



Let P be a point within the curve, PQ a fixed line. Let AB be any line crossing the curve, the normal to which, Pp, makes an angle θ with PQ, and let the intercepted chord AB $= u$. Let a second line, whose normal makes an angle θ', be drawn intersecting AB; the number of such lines is $\pm u \sin(\theta - \theta')$ according as θ is greater or less than θ': the values of θ, θ' to range from 0 to π. The number of lines then that intersect AB is

$$u\int_0^\theta \sin(\theta-\theta')\,d\theta' + u\int_\theta^\pi \sin(\theta'-\theta)\,d\theta' = u(1-\cos\theta) + u(1+\cos\theta),$$

or $2u$. If $Pp = p$, the number of lines intersecting all lines parallel to AB is

$$2\int u\,dp = 2A,$$

where A is the area of the curve; and consequently the total number of pairs of lines that intersect is $2\pi A$. It remains to find the total number of lines that cross the curve. Let p, p' be the perpendiculars from P upon a pair of parallel tangents to the curve, then $p+p'$ is the number of lines which cross parallel to those tangents, and the total number is therefore

$$\int_0^{\pi} (p+p')\,d\theta = P,$$

where P is the perimeter of the curve. The required probability of intersection is, therefore, $\dfrac{2\pi A}{P^2}$.

2773. (Proposed by MATTHEW COLLINS, B.A.)—When a given angle rolls upon a fixed parabola, the locus of its vertex is well known to be a hyperbola having the same focus and directrix as the parabola. Conversely, when a given parabola rolls within a fixed angle, show that its focus and vertex describe lines of the fourth and sixth orders respectively; find the *actual equations* of both these curves, and thence show that the vertex of the fixed angle is a conjugate point on the former curve, and that the equation of the latter, when the given angle is a right angle, becomes

$$x^2y^2(x^2+y^2+3a^2) = a^6 \quad \text{or} \quad x^{\frac{2}{3}}y^{\frac{2}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}}) = a^2,$$

the sides of the given angle being the axes, and a equal to the distance of the focus from the vertex.

Solution by the Rev. J. WOLSTENHOLME, M.A.

The equation of a parabola touching the axes of x, y, which are inclined at angle ω, being $\left(\dfrac{x}{h}\right)^{\frac{1}{2}} + \left(\dfrac{y}{k}\right)^{\frac{1}{2}} = 1$, the coordinates of the focus are

$$\frac{X}{k} = \frac{Y}{h} = \frac{hk}{h^2+k^2+2hk\cos\omega},$$

and the equation of the directrix is

$$x(h+k\cos\omega) + y(k+h\cos\omega) = hk\cos\omega.$$

Hence, if $4a$ be the latus rectum, we have

$$2a = \frac{\dfrac{hk^2(h+k\cos\omega)+h^2k(k+h\cos\omega)}{h^2+k^2+2hk\cos\omega} - hk\cos\omega}{\left\{(h+k\cos\omega)^2+(k+h\cos\omega)^2-2(h+k\cos\omega)(k+h\cos\omega)\cos\omega\right\}^{\frac{1}{2}}}$$

$$= \frac{2h^2k^2\sin^2\omega}{(h^2+k^2+2hk\cos\omega)^{\frac{3}{2}}};$$

whence the equation of the locus of the focus is

$$a^2(X^2+Y^2+2XY\cos\omega) = X^2Y^2\sin^2\omega,$$

a curve of the fourth degree having a conjugate circular point at the origin, and having the four asymptotes $X = \pm a \operatorname{cosec}\omega$, $Y = \pm a \operatorname{cosec}\omega$. The vertex is most readily found from being the point of contact of a tangent parallel to the directrix, which gives at once, if (X, Y) be the vertex,

$$\frac{(kY)^{\frac12}}{h+k\cos\omega} = \frac{(hX)^{\frac12}}{k+h\cos\omega} = \frac{hk}{h^2+k^2+2hk\cos\omega} = \frac{a}{\sin^2\omega}\,\frac{(h^2+k^2+2hk\cos\omega)^{\frac12}}{hk};$$

three equations from which to eliminate h, k. In the case where $\omega = \frac12\pi$, we have

$$\frac{X}{k^3} = \frac{Y}{h^3} = \frac{hk}{(h^2+k^2)^2} = \frac{a^2(h^2+k^2)}{h^3k^3} \quad\text{or}\quad hX^{\frac14} = kY^{\frac14} = (X^{\frac23}+Y^{\frac23})^2,$$

and $$a^2 = \frac{k^3Y}{h^2+k^2} = \frac{(X^{\frac23}+Y^{\frac23})^6}{(X^{\frac23}+Y^{\frac23})^4(X^{-\frac23}+Y^{-\frac23})} = X^{\frac23}Y^{\frac23}(X^{\frac23}+Y^{\frac23}),$$

or $$a^6 = X^2Y^2(X^2+Y^2+3a^2).$$

2774. (Proposed by J. J. WALKER, M.A.) — A central conic $(b^2x^2 \pm a^2y^2 = a^2b^2)$ is turned in its own plane about its centre through a right angle; prove that the locus of the intersection of the normal at any point on the given conic with the same line in its new position is one of the two sextic curves

$$(x^2+y^2)^2\left\{a^2(x\pm y)^2 \pm b^2(x\mp y)^2\right\} = (a^2\mp b^2)^2(x^2-y^2)^2.$$

Quaternion Solution by W. H. LAVERTY, B.A.

The vector equations to the two normals are

$$\rho = \alpha\cos\tau + \beta\sin\tau + x(b^2\alpha\cos\tau + a^2\beta\sin\tau),$$

and $$\rho' = \frac{a}{b}\beta\cos\tau - \frac{b}{a}\alpha\sin\tau + y(ba\beta\cos\tau - ab\alpha\sin\tau);$$

and that these may coincide, we have

$$-x\,.\,ab(a^2\sin^2\tau + b^2\cos^2\tau) = ab + (b^2-a^2)\sin\tau\cos\tau.$$

Whence, if $a^2\sin^2\tau + b^2\cos^2\tau = P$, the equation to the locus becomes

$$ab.P.\rho = \sin\tau.\cos\tau(a^2-b^2)\left\{b.\alpha(a\sin\tau+b\cos\tau) + a.\beta(a\sin\tau - b\cos\tau)\right\}.$$

To transform this to a Cartesian equation, we have

$$P.x = \sin\tau.\cos\tau(a^2-b^2)(a\sin\tau+b\cos\tau),$$
$$P.y = \sin\tau.\cos\tau(a^2-b^2)(a\sin\tau-b\cos\tau);$$

therefore $$P^2\left\{a^2(x-y)^2 + b^2(x+y)^2\right\} = 4a^2b^2\sin^2\tau\cos^2\tau(a^2-b^2)^2,$$
$$(x^2+y^2)^2P^2 = 4\sin^4\tau\cos^4\tau(a^2-b^2)^4,$$
$$(x^2-y^2)^2P^4 = 16a^2b^2\sin^6\tau\cos^6\tau(a^2-b^2)^4,$$

whence the required equation. Similarly for the hyperbola.

2244. (Proposed by the late W. LEA.)—Form 11 symbols into sets, 5 symbols in a set, so that every combination of 4 symbols shall appear once in the sets.

2712. (Proposed by the late W. LEA.)—Form 15 symbols into sets, 5 symbols in each set, so that every combination of 4 symbols shall appear once in the sets.

2755. (Proposed by the late W. LEA.)—Form 16 symbols into sets, 4 symbols in each set, so that every triad in the symbols shall appear once in the sets.

Solution by the Rev. T. P. KIRKMAN, M.A., F.R.S.

The problem of 4-plets exhausting the triads of 16 is a case of the more general theorem which I gave in the *Cambridge and Dublin Mathematical Journal*, 1853, p. 42, thus: "2^n young ladies can all walk together in fours, day by day till every three have walked together." The proof is very simple. If $n = 2m$, we take for our 4^m symbols the repeating variations that can be made of them in 4-plets. Then every triplet is completed into a 4-plet by the rule that no 4-plet shall have only three rth places either like or unlike. Thus the triplet *abbadd.bbcaab.cdaaaa* ($m=6$) is completed into a 4-plet of sixes by the six *dddadc*. When $n = 2m+1$, we join to the 4^m symbols made with *abcd* the 4^m made with abcd, and add this rule, that no 4-plet shall have only three Italic or only three Roman symbols; thus, if out of 512 young ladies the three (abbd), (*abbb*), and (*adaa*) choose to walk together on any day, their companion must be (adac), and I have shown that any triplet whatever will determine the entire arrangement for the day.

To form 5-plets with eleven elements so as once to exhaust the 4-plets (Question 2244), we first form the 12 5-plets that contain the two elements $0a$; thus

$$\begin{matrix} 0a123 & 0a147 & 0a159 & 0a168 & \\ 0a456 & 0a258 & 0a267 & 0a249 & \quad (0a) \\ 0a789 & 0a369 & 0a348 & 0a357 & \end{matrix}$$

which are obtained by properly reading any one of the four squares of triplets in figures. We call this set of 12 $(0a)$. There must be in the solution a set of 12 (01), another (02), &c., another (12), &c., such that, if we form them all, we shall construct all the 11.6 quintuplets 10 times. Denote by $(a1, 4678)(0a)$ the operation on $(0a)$ with the two circles $a1$, 4678, that is, the exchange of a and 1, and the putting 6 for 4, 7 for 6, 8 for 7, and 4 for 8. Then we find that

$$\begin{aligned} (01) &= (a1, 4678)(0a), & (a1) &= (01, 8764)(0a), \\ (02) &= (a2, 5489)(0a), & (a2) &= (02, 9845)(0a), \\ (03) &= (a3, 6597)(0a), & (a3) &= (03, 7956)(0a), \\ (07) &= (a7, 1345)(0a), & (a4) &= (04, 2197)(0a), \\ (08) &= (a8, 2156)(0a), & (a5) &= (05, 3278)(0a), \\ (09) &= (a9, 3264)(0a), & (a6) &= (06, 1389)(0a), \\ (04) &= (a4, 7912)(0a), & (a7) &= (07, 5431)(0a), \\ (05) &= (a5, 8723)(0a), & (a8) &= (08, 6512)(0a), \\ (06) &= (a6, 9831)(0a), & (a9) &= (09, 4623)(0a). \end{aligned}$$

We have now every quintuplet containing 0 and all containing a, in number $12+18+18 = 48$.

If $\theta = (4678)$, $\theta^{-1} = \theta^3 = (8764)$, and $\theta^2 = (47, 68)$; and the operation

(0*a*, 47, 68) on (01) will give (*a*1), and *vice versâ*. (10) and (1*a*) can be formed by reading horizontally, vertically, and parallel to the diagonals, the two triplet squares in the right members following:

$$(10) = 10\,.\,\begin{matrix} a23 \\ 657 \\ 849 \end{matrix} \qquad (1a) = 1a\,.\,\begin{matrix} 023 \\ 854 \\ 679 \end{matrix}$$

and we obtain 1*a*, 12, 13, &c. from (10), thus

$$\begin{aligned} (1a) &= (0a, 47, 68)(10), & (16) &= (06, 29, 8a)(10), \\ (12) &= (02, 96, 54)(10), & (15) &= (05, 38, 42)(10), \\ (13) &= (03, 85, 79)(10), & (17) &= (07, a4, 93)(1a), \end{aligned}$$

$$\begin{aligned} (18) &= (08, 53, a6)(10), \\ (14) &= (04, 7a, 25)(10), \\ (19) &= (09, 62, 37)(10). \end{aligned}$$

We have now every quintuplet containing 1. Next we write, by (2*a*) = (0*a*, 94, 85) (20),

$$(20) = 20\,.\,\begin{matrix} 1a3 \\ 846 \\ 795 \end{matrix} \qquad (2a) = 2a\,.\,\begin{matrix} 103 \\ 596 \\ 748 \end{matrix}$$

and we obtain, omitting (2*a*) and (21) already found,

$$\begin{aligned} (23) &= (03, 56, 47)(20), & (27) &= (07, 81, 34)(20), \\ (28) &= (08, 17, 5a)(20), & (29) &= (09, 4a, 16)(20), \\ (24) &= (04, a9, 73)(20), & (25) &= (05, 63, a8)(20). \\ (26) &= (06, 35, 91)(20), \end{aligned}$$

This gives us every 5-plet containing 2. Next, we write (30) and (3*a*) = (0*a*, 69, 57) (30),

$$(30) = 30\,.\,\begin{matrix} 12a \\ 495 \\ 687 \end{matrix} \qquad (3a) = 3a\,.\,\begin{matrix} 120 \\ 467 \\ 985 \end{matrix}$$

The process is continued thus:—

$$\begin{aligned} (34) &= (04, 27, 61)(30), & (36) &= (06, 9a, 14)(30), \\ (39) &= (09, a6, 82)(30), & (38) &= (08, 51, 29)(30), \\ (35) &= (05, 18, 7a)(30), & (37) &= (07, 42, a5)(30), \end{aligned}$$

which gives all containing 3.

We now add the 54 following to the 12 above formed with 0*a*:—

01657	02795	*a*1854	*a*2748	12947	17958
01849	02347	*a*1679	*a*2397	12856	23875
01254	02365	*a*1257	*a*2368	12376	23496
01379	02389	*a*1349	*a*2345	12348	24765
01358	03495	*a*1356	*a*3467	12359	24589
01629	03687	*a*1829	*a*3958	13547	26789
01278	04587	*a*1246	*a*4759	13689	34789
01346	04679	*a*1378	*a*4986	14569	34568
02846	05689	*a*2596	*a*5876	14867	35679

If it is possible to form quintuplets with 15 letters so as to exhaust the quadruplets, there must be 26 containing the final letters *de*. These are easily formed by cyclically permuting the 13 elements 123 .... 90*abc* under 125 and 138. This gives us

(*de*) = *de*125, *de*236, ... *de*138, *de*249 .....

There must be a set (*d*1) of 26, and another (*e*1). We readily find

$$\begin{aligned} (1e) &= (1d, 2358, 7a90)(de), \\ (1d) &= (1e, 8532, 09a7)(de), \end{aligned}$$

results so far analogous to the preceding.

But I cannot find the substitutions that will correctly give (2*e*), (2*d*), &c. I shall be agreeably surprised to see this problem solved. It would be pleasant to find it to be a simple evolution of the famous fifteen young ladies whom I had the honour of introducing to the planet for the first time in the *Lady's and Gentleman's Diary* for 1850. To my original puzzle of seven arrangements of five triads so that every pair should once walk abreast, Professor SYLVESTER added a second, to make the school of 15 walk out every day in the quarter in 13.7 columns of triplets, till every three have once walked abreast. I gave, at the time, one solution of Prof. SYLVESTER'S question in the *Cambridge and Dublin Mathematical Journal.* It would be a memorable sight to see his 13.7 columns of 5 triplets made to perform the right-face, so as to present the 13.7.3 quintuplets here required. Any field-marshal could transform by a word the column of 5.3 into a dangerous phalanx of 3.5; but I fear that a couple of generalissimos would find it a heavy strain on their tactics to turn out day by day for the quarter the exact quintuplets of Mr. LEA.

2893. (Proposed by M. W. CROFTON, F.R.S.)—If there be (n) quantities a, b, c, d, &c., each of which takes independently a given number of values $a_1a_2a_3 \ldots b_1b_2b_3 \ldots$ &c. (the number may be different for each), if we put

$$\Sigma(a) = a+b+c+d+\&c.,$$

and if for shortness we denote "the mean value of x" by $M(x)$, prove that

$$M[\Sigma(a)] = M(a)+M(b)+M(c)+\&c. = \Sigma[M(a)],$$
$$M[\Sigma(a)]^2 = \{\Sigma[M(a)]\}^2 - \Sigma[M(a)]^2 + \Sigma[M(a^2)].$$

Solution by W. H. H. HUDSON, M.A.

Here a may have any of r values, $a_1, a_2, a_3 \ldots$; b any of s values, $b_1, b_2, b_3 \ldots$; c any of t values, $c_1, c_2, c_3 \ldots$; &c. &c.;

therefore $a+b+c+\ldots$ may have any of the $rst\ldots$ values

$$a_1+b_1+c_1+\ldots, \quad a_2+b_2+c_2+\ldots, \quad \ldots\ldots,$$

whereof a_1 occurs in $st\ldots$ sets, b_1 in $rt\ldots$ sets, and so on;

$$\text{therefore } \frac{\Sigma(a+b+c)\ldots}{rst\ldots} (= M\Sigma a) = \frac{st\ldots(a_1+a_2+\ldots)+t\ldots(b_1+b_2+\ldots)+\ldots}{rst\ldots}$$

$$= \frac{a_1+a_2+\ldots}{r} + \frac{b_1+b_2+\ldots}{s} + \ldots = Ma+Mb+Mc+\ldots = \Sigma Ma.$$

Again, $(\Sigma a)^2$ may have $rst\ldots$ values, of which the type is

$$\ldots a^2+b^2+c^2+\ldots+2ab+2bc+\ldots$$

a_1^2 will occur in $st\ldots$, b_1^2 in $rt\ldots$, and so on; a_1b_1 will occur in $t\ldots$, b_1c_1 in $r\ldots$, and so on

Therefore $M(\Sigma a)^2 =$

$$\frac{st\ldots(a_1^2+a_2^2+\ldots)+rt\ldots(b_1^2+\ldots)+2t\ldots(a_1b_1+\ldots)+2r\ldots(b_1c_1+\ldots)}{rst}$$

$$= \frac{a_1^2+a_2^2+\ldots}{r} + \frac{b_1^2+b_2^2+\ldots}{s} + \ldots + \frac{2(a_1b_1+a_2b_2+\ldots)}{rs} + \frac{2(b_1c_1+\ldots)}{st} + \ldots$$

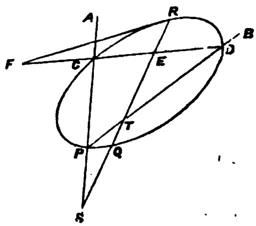
Now $$\left(\frac{a_1+a_2+\dots}{r}+\frac{b_1+b_2+\dots}{s}+\frac{c_1+c_2+\dots}{t}+\dots\right)^2$$
$$-\left\{\left(\frac{a_1+a_2+\dots}{r}\right)^2+\left(\frac{b_1+b_2+\dots}{s}\right)^2+\dots\right\}$$
$$=\frac{2(a_1b_1+\dots)}{rs}+\frac{2(b_1c_1+\dots)}{st}+\dots;$$

therefore $$\mathrm{M}(\Sigma a)^2=\mathrm{M}(a^2)+\mathrm{M}(b^2)+\dots$$
$$+\left\{\left(\frac{a_1+a_2+\dots}{r}\right)+\left(\frac{b_1+b_2+\dots}{s}\right)+\dots\right\}^2$$
$$-\left(\frac{a_1+a_2+\dots}{r}\right)^2-\left(\frac{b_1+b_2+\dots}{s}\right)^2-\dots$$
$$=\Sigma\mathrm{M}a^2+(\Sigma\mathrm{M}a)^2-\Sigma(\mathrm{M}a)^2.$$

2887. (Proposed by ARCHER STANLEY.)—1. If a point P on a conic be connected with any two fixed points A and B in its plane, all chords which are divided harmonically by PA and PB will be concurrent.

2. The locus of the point of concurrence when P is variable is another conic which has double contact with the given one on AB.

Solution by F. D. THOMSON, M.A.



1. Let QR be the chord divided harmonically by PA, PB. Then, constructing as in the figure,
$$-1=\mathrm{P}\{\mathrm{SQTR}\}=\mathrm{P}\{\mathrm{CQDR}\}$$
$$=\mathrm{R}\{\mathrm{CEDF}\};$$
therefore F is the pole of QR, and therefore QR passes through the pole of CD, *i.e.* through a fixed point.

2. If O be the pole of CD, then when P varies the locus of O is a conic. For consider the points in which the locus of O is cut by the tangent to the given conic through A. Let the tangent through A have L for its point of contact. Let BL meet the conic in M, AM meet the conic in N. Then the poles of LM and LN are the points on the required locus upon the tangent AL. Hence the locus is a conic. Also the tangents to the given conic at the points where AB meets it are seen to be tangents to the required locus. Hence the locus is a conic having double contact with the given conic at two points on AB.

The particular case when A and B are the circular points is given by Salmon, p. 257.

2728. (Proposed by W. S. McCay, B.A.) — Given three planes and their poles with regard to a system of quadrics, the locus of centre is a right line.

Solution by the PROPOSER.

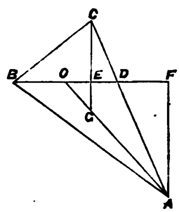
For the lines joining the vertices of a tetrahedron to the corresponding vertices of its polar tetrahedron belong to the same system of generators of an hyperboloid of one sheet. (Salmon's *Geometry of Three Dimensions*, p. 179.) Hence, the centre being the pole of the plane at infinity, we see that if through each of the given poles we draw lines parallel to the intersection of the other two planes, the hyperboloid of which these are generators passes through the intersection of the three planes, and the locus of centre is the generator of the same system at that point.

2597. (Proposed by W. H. H. Hudson, M.A.)—A right cone, whose weight may be neglected, is suspended from a point in its rim; it contains as much fluid as it can: show that the whole pressure upon its surface is

$$\tfrac{1}{3}\pi\rho g h^3 \frac{\sin\alpha\cos\theta}{\cos^2\alpha}\left\{\frac{\cos(\theta+\alpha)}{\cos(\theta-\alpha)}\right\}^{\frac{3}{2}},$$

where h and 2α are the height and vertical angle of the cone, and θ is determined from $3\sin 2\theta = 4\sin 2(\theta-\alpha)$.

Solution by STEPHEN WATSON.



Let C be the point of suspension; ABC a vertical section of the cone; BF a horizontal line cutting AC in D, and meeting a perpendicular from A in F; and O the middle of BD. Join AO, and take $OG = \frac{1}{4}OA$; then G is the centre of gravity of the fluid; hence CG must cut OF perpendicularly in E, so that $OE = \frac{1}{4}OF$. Put $\angle CBD = \theta$, the area of the horizontal surface of the fluid $=$ S, and the area of the surface of the cone in contact with the fluid $\doteq$ (A).

Now $\quad BE = BC\cos\theta = 2h\tan\alpha\cos\theta,$

$$BF = BA\cos(\tfrac{1}{2}\pi-\alpha-\theta) = h\sec\alpha\sin(\theta+\alpha),$$

and $\quad BO = \frac{1}{2}BD = \dfrac{BC\sin BCD}{2\sin BDC} = \dfrac{h\sin\alpha}{\cos(\theta-\alpha)};$

hence, by substitution in $BF-BO = 4(BE-BO)$ or $BF+3BO = 4BE$, we have $\sin(\theta+\alpha)\cos(\theta-\alpha)+3\sin\alpha\cos\alpha = 8\sin\alpha\cos\theta\cos(\theta-\alpha)$,

$$\therefore\quad \tfrac{1}{2}(\sin 2\theta+\sin 2\alpha)+\tfrac{3}{2}\sin 2\alpha = 4\sin\alpha\left\{\cos(2\theta-\alpha)+\cos\alpha\right\}$$
$$= 2\left\{\sin 2\theta-\sin 2(\theta-\alpha)\right\}+2\sin 2\alpha,$$

therefore $3\sin 2\theta = 4\sin 2(\theta-\alpha)$, the condition for finding θ.

Again, if a, b $(a = OB)$ be the semi-axes of the surface S, we have, by

putting $\frac{1}{2}\pi - \alpha + \theta$ for α, and α for β, in equation (12), p. 295, Vol. II., of Davies' edition of Hutton's *Course*,

$$S = \pi ab = \frac{\pi BO^2}{\cos\alpha}\left\{\cos(\theta+\alpha)\cos(\theta-\alpha)\right\}^{\frac{1}{2}} = \frac{\pi h^2 \sin^2\alpha}{\cos\alpha}\cdot\frac{\cos^{\frac{1}{2}}(\theta+\alpha)}{\cos^{\frac{3}{2}}(\theta-\alpha)}.$$

Also the distance of the centre of gravity of (A) from the horizontal line BF is $\quad \frac{1}{3}AF = \frac{1}{3}BA\sin(\frac{1}{2}\pi - \alpha - \theta) = \frac{1}{3}h\sec\alpha\cos(\theta+\alpha)$ .... (1).

Moreover, the orthogonal projections of the surfaces S and (A) upon the base of the cone must be equal, that is,

$$(A)\sin\alpha = S\cos\theta, \quad \text{therefore} \quad (A) = \frac{\cos\theta}{\sin\alpha}S\,;$$

hence the whole pressure upon the cone's surface is

$$\tfrac{1}{3}\rho g \times (A) \times (1) = \tfrac{1}{3}\pi\rho g h^3 \frac{\sin\alpha\cos\theta}{\cos^2\alpha}\left\{\frac{\cos(\theta+\alpha)}{\cos(\theta-\alpha)}\right\}^{\frac{3}{2}}.$$

2108. (Proposed by W. K. Clifford, B.A.) — Required analogues in Solid Geometry to the following propositions in Plane Geometry:—

(*a*.) The perpendiculars of a triangle meet in a point.

(*b*.) The middle points of the diagonals of a quadrilateral are in one straight line.

(*c*.) The circles whose diameters are the diagonals of a quadrilateral have a common radical axis.

(*d*.) Every rectangular hyperbola circumscribing a triangle passes through the intersection of perpendiculars.

(*e*.) Every rectangular hyperbola to which a triangle is self-conjugate passes through the centres of the four touching circles.

(*f*.) $\qquad \sin(A+B) = \sin A\cos B + \cos A\sin B.$

(*g*.) The sum of the angles of a triangle = two right angles.

(*h*.) In any triangle $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{C}$.

Solution by the Proposer.

I have an analogue for each of the four (*d*), (*e*), (*f*), (*g*), and more than one for each of the others obtained by extensions of Mr. Greer's methods. *E. g.*:—

(*c*.) A straight line cuts the faces of a tetrahedron ABCD in *a*, *b*, *c*, *d*; the spheres whose diameters are A*a*, B*b*, C*c*, D*d*, have a common radical axis. Hence the middle points of these four lines are in one plane.

Let a conicoid whose asymptotic cone has three generating lines at right angles be called a rectangular conicoid.

(*d*), (*e*). Every rectangular conicoid circumscribing a tetrahedron whose perpendiculars meet in a point, passes through the point. And every rectangular conicoid to which a tetrahedron is self-conjugate, passes through the centres of the eight touching spheres.

(*f*.) $\sin(\mathrm{ABC}) = \sin(\mathrm{ABD}+\mathrm{BCD}+\mathrm{CAD})$

$$= \sin(\mathrm{BCD}) . \cos\widehat{\mathrm{AD}} + \sin(\mathrm{CAD}) . \cos\widehat{\mathrm{BD}} + \sin(\mathrm{ABD}) . \cos\widehat{\mathrm{CD}},$$

where A, B, C, D are four lines in space, and

$$\sin^2 \mathrm{ABC} = \begin{vmatrix} 1, & \cos\widehat{\mathrm{AB}}, & \cos\widehat{\mathrm{AC}} \\ \cos\widehat{\mathrm{AB}}, & 1, & \cos\widehat{\mathrm{BC}} \\ \cos\widehat{\mathrm{AC}}, & \cos\widehat{\mathrm{BC}}, & 1, \end{vmatrix}$$

(*g*.) In the triangle case this should be written

$$(\mathrm{BC})+(\mathrm{CA})+(\mathrm{AB}) = 0.$$

The analogue is then obviously

$$(\mathrm{BCD})-(\mathrm{CDA})+(\mathrm{DAB})-(\mathrm{ABC}) = 0,$$

A, B, C, D being any four planes.

(*h*.) In any tetrahedron,

$$\frac{\mathrm{AC}.\mathrm{DB}}{\sin\widehat{\mathrm{AC}}.\sin\widehat{\mathrm{DB}}} = \frac{abcd}{\mathrm{V}^2} = \frac{a}{\cos \mathrm{A}} = \frac{\mathrm{V}^6}{(abcd)^2} . \cos \mathrm{A} \cos \mathrm{B} \cos \mathrm{C} \cos \mathrm{D},$$

where a, b, c, d are the faces, and $\cos^2 \mathrm{A} = \begin{vmatrix} 1, & \cos\widehat{\mathrm{BC}}, & \cos \mathrm{BD} \\ \cos\widehat{\mathrm{BC}}, & 1, & \cos\widehat{\mathrm{CD}} \\ \cos\widehat{\mathrm{BD}}, & \cos\widehat{\mathrm{CD}}, & 1 \end{vmatrix}$

($\widehat{\mathrm{AB}}$, &c. denoting angles between planes).

2651. (Proposed by W. S. Burnside, M.A.)—Determine the form of the solution of the differential equation

$$2f(x)\frac{d^2\mathrm{U}}{du^2} + 3f'(x)\frac{d\mathrm{U}}{du} + \left\{f''(x) \pm n^2\right\} \mathrm{U} = 0 \quad \text{........} \quad (1).$$

Solution by the Proposer.

The differential equation

$$f(x)\frac{d^2y}{du^2} + \tfrac{1}{2}f'(x)\frac{dy}{du} \pm n^2 y = 0 \quad \text{..............} \quad (2),$$

is reduced to the form $\dfrac{d^2y}{dt^2} \pm n^2 y = 0$ by assuming $t = \displaystyle\int \frac{du}{\sqrt{f(x)}}$ for the independent variable; whence, taking the lower sign,

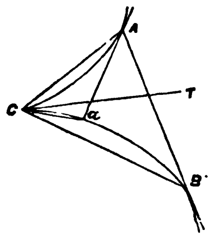
$$y = ce^{n\int \frac{du}{\sqrt{f(x)}}} + c_1 e^{-n\int \frac{du}{\sqrt{f(x)}}}.$$

Again, differentiating the equation (2) with regard to x, and writing U for $\dfrac{dy}{dx}$, and $2f(x)$ for $f(x)$, we find the given equation (1), the solution of which is therefore

$$\mathrm{U} = (fx)^{-\frac{1}{2}} \left\{ ae^{n\int \frac{dx}{\sqrt{2f(x)}}} + be^{-n\int \frac{dx}{\sqrt{2f(x)}}} \right\}.$$

2895. (Proposed by J. J. Walker, M.A.)—At any point on a cusped cubic a tangent is drawn meeting the cubic in a second point; from the first point a line is drawn touching the cubic in a third point. Prove that the lines drawn from the cusp to these points form with the cuspidal tangent a harmonic pencil.

Solution by W. H. H. Hudson, M.A.



Let the tangent at A meet the cubic in a, and from A draw AB tangent to the cubic. Let C be the cusp, CT the tangent thereat. It is required to prove that CB, Ca, CT, CA form an harmonic pencil. Take ABC as triangle of reference. The equation of the cubic is $\gamma(l\alpha+m\beta)^2+n\alpha^2\beta=0$.

[This may be obtained by taking the complete equation of 10 terms, and simplifying it from the considerations (1) that the cubic passes through A, B, C, (2) AB touches at B, (3) any line through C meets it in two points, then (4) the two lines that meet it in three points at C are coincident.]

$l\alpha+m\beta=0$ is the equation of CT,

$l^2\gamma+n\beta=0$ is the equation of the tangent at A (Aa),

and it meets the cubic where $2l\alpha+m\beta=0$, which is therefore the equation of CA. From the form of these equations we see that CB, Ca, CT, CA form an harmonic pencil.

[Mr. Thomson and the Proposer remark that the theorem follows at once from Salmon's *Higher Plane Curves*, Art. 178.]

2902. (Proposed by J. J. Walker, M.A.)—The circle passing through three points on a parabola, the normals at which co-intersect, always passes through the vertex; and if the point of co-intersection of the normals describe a coaxial conic having the vertex as centre and axes in the ratio of $m:n$, the locus of the centre of the circle will be a conic having the focus as centre, and axes in the ratio of $2m:n$.

Solution by the Proposer; W. Roberts; R. Tucker, M.A.; *and others.*

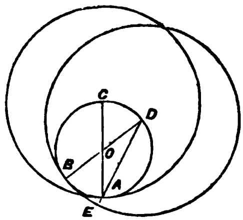
Let (α, β) be the common point of the normals, and $y^2=4ax$ the parabola; then the equation to the circle is readily shown to be

$$x^2+y^2-(\alpha+2a)x-\tfrac{1}{4}\beta y=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

From this equation we see that the circle passes through the vertex; also that if (x', y') be the centre, $2x'=\alpha+2a$, $4y'=\beta$. Let $n^2\alpha+m^2\beta^2=k^2$ be the locus of (α, β), then substituting $2(x'-a)$ for α and $4y'$ for β, $n^2x'^2+4m^2y'^2=\tfrac{1}{4}k^2$ is the locus of (x', y')—a conic having its centre at $(-a, 0)$ and axes in the ratio of $2m:n$.

2898. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—**A circle rolls with internal contact on a fixed circle of half its radius; prove that the envelope of any chord of the rolling circle is a circle which reduces to a point for a diameter.**

Solution by W. H. H. HUDSON, M.A.; F. D. THOMSON, M.A.; R. TUCKER, M.A.; *and others.*



Let the diameter considered be the line joining the centres when the point of contact is A; when B is the point of contact let D be the centre of the large circle. Join DA; and produce it to meet the large circle in E.

Since BOA = 2ODA, arc BA = arc BE; therefore E is the point to which A has rolled, and DE is the new position of AC. Hence the diameter always passes through A.

Now conceive a chord parallel to this diameter. It is always at a constant distance from it. Hence, if from A we draw a perpendicular upon it, this perpendicular is constant. It therefore envelopes a circle with centre A, which degenerates to the point A when the chord becomes a diameter.

2875. (Proposed by the REV. J. WOLSTENHOLME, M.A.)—Any tangent to a conic is, of course, divided in involution by three other tangents, and the lines joining their points of intersection to one of the foci of the conic; prove that the distance between the double points of the involution subtends a right angle at the focus of the conic. The locus of these double points, the three tangents being fixed, is a cubic having a double point at the focus, whose nature I have not examined.

Solution by ARCHER STANLEY.

Let A, B, C be fixed tangents to the given conic, and X a variable one; moreover, let P and P′ be the tangents from a given point p. (In the question p is a focus, P and P′ therefore pass through the circular points at infinity, and all harmonic conjugates relative to them are orthogonal.) Now, by a well known theorem, the rays from p to the three pairs of opposite intersections of the quadrilateral ABCX are in involution, and to this involution likewise belong all pairs of tangents from p to the several conics inscribed in the same quadrilateral. Now the given conic obviously belongs to this series, hence P, P′ form a pair of conjugate rays of the involution; the double rays D_1, D_2 thereof must, consequently, be harmonic conjugates relative to P and P′, no matter what position X may have. This proves the first part of the theorem.

With respect to the locus of the intersections of X with the double rays D_1 and D_2, I observe that the latter are the tangents at p to the two conics inscribed in ABCX which pass through p. Hence, not only does X determine the harmonic conjugates D_1, D_2 relative to P and P′, but, conversely, every

such line-pair D_1, D_2 determines its corresponding tangent X; the latter is, in fact, the fourth tangent common to the given conic, and to that which is inscribed in ABC so as to touch D_1 (or D_2) at p. It is easy, therefore, to determine the order of the locus, for through any point ξ of an arbitrary line L will pass one line-pair (D_1, D_2) corresponding to one tangent X, which cuts D say in x. Again, two tangents X pass through every point x; and to them will correspond two line-pairs (D_1, D_2) cutting L in four points ξ. Hence, between the points ξ and x there is a (1, 4) correspondence, and consequently there are five points in which x and ξ are coincident. It is easy to see, however, that one of the latter is always on P and another on P′; deducting these, which are not proper points of the required locus, we conclude that the latter is a cubic, of which p is a double point and P, P′ the tangents thereat, since on every line through p there is but one point of the locus which is not coincident therewith, and on P (and P′) there are none. (When p is a focus of the given conic, it is also a conjugate point on the cubic.) It is easy to find the nine intersections of the cubic with the fixed tangents A, B, C. Three of them will be the points a, b, c, where the sides A, B, C are intersected by the connector of p with the respective opposite vertices (BC), (CA), (AB); for instance, a is obviously a double point of the involution on the second tangent X which passes through it. The remaining two points of the cubic on A may be best found by conceiving X to coincide with it, or, in other words, to intersect it in its point of contact α; for then a and α will be one pair of conjugate points of the involution on A, another pair will be the intersections (AB), (AC). The points where A cuts the cubic, therefore, will be those which divide both the segments $\overline{a\alpha}$ and $\overline{(AB)\,(AC)}$ harmonically. In this manner the *order* of the locus might have been more readily determined.

2003. (Proposed by W. H. Laverty.)—If A′, B′, C′, D′, E′, F′ be the minors which are the coefficients of A, B, C, D, E, F, respectively, in the different expansions of

$$\begin{vmatrix} A & F & E \\ F & B & D \\ E & D & C \end{vmatrix}; \text{ and if } \phi_1 \text{ denote either } (A', B', C', D', E', F')(\lambda, \mu, \nu)^2, \text{ or } (A', B', C', D', E', F')(a, b, c)^2;$$

$$\text{if } \phi_2 = (1, 1, 1, -\cos A, -\cos B, -\cos C)(\lambda, \mu, \nu);$$

if also $P = \left(\dfrac{d\phi_2}{d\mu}\right)\left(\dfrac{d\phi_1}{d\nu}\right) - \left(\dfrac{d\phi_1}{d\mu}\right)\left(\dfrac{d\phi_2}{d\nu}\right)$; and Q and R denote similar quantities involving ν, λ, and λ, μ, respectively; then show that, if a focus of the conic $(A, B, C, D, E, F)(\alpha, \beta, \gamma)^2 = 0$ be known to be in the line $\lambda\alpha + \mu\beta + \nu\gamma = 0$, its coordinates (λ', μ', ν') are proportional to the minors of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ \lambda & \mu & \nu \\ P & Q & R \end{vmatrix}$, and the equation to the diameter of the conic is

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \lambda' & \mu' & \nu' \\ \left(\dfrac{d\phi_1}{da}\right) & \left(\dfrac{d\phi_1}{db}\right) & \left(\dfrac{d\phi_1}{dc}\right) \end{vmatrix} = 0.$$

Solution by the PROPOSER.

It is evident that if we find the pole of $\lambda\alpha+\mu\beta+\nu\gamma=0$, and from this pole draw a perpendicular to the line, this perpendicular will meet the line in the focus. The coordinates, then, of the pole of the line are

$$\frac{\alpha'}{\left(\frac{d\phi_1}{d\lambda}\right)}=\frac{\beta'}{\left(\frac{d\phi_1}{d\mu}\right)}=\frac{\gamma'}{\left(\frac{d\phi_1}{d\nu}\right)}\ \ldots\ldots\ldots\ldots\ldots\ldots(1);$$

and the condition that a line $L\alpha+M\beta+N\gamma=0$ ....(2)
may be perpendicular to $\lambda\alpha+\mu\beta+\nu\gamma=0$, we should find to be

$$L(\lambda-\mu\cos C-\nu\cos B)-M(\mu-\nu\cos A-\lambda\cos C)+N(\nu-\lambda\cos B-\mu\cos A)=0,$$

or
$$L\left(\frac{d\phi_2}{d\lambda}\right)+M\left(\frac{d\phi_2}{d\mu}\right)+N\left(\frac{d\phi_2}{d\nu}\right)=0\ \ldots\ldots\ldots\ldots(3).$$

Also we have, from (1), since $(\alpha'\ \beta'\ \gamma')$ is on the line,

$$L\left(\frac{d\phi_1}{d\lambda}\right)+M\left(\frac{d\phi_1}{d\mu}\right)+N\left(\frac{d\phi_1}{d\nu}\right)=0\ \ldots\ldots\ldots\ldots(4);$$

therefore, from (2), (3), (4), we get (by eliminating L, M, N) for the equation of the perpendicular $\qquad P\alpha+Q\beta+R\gamma=0$ ....(5),
and for the intersection of this with $\lambda\alpha+\mu\beta+\nu\gamma=0$ we have (for the coordinates of the focus)

$$\frac{\lambda'}{\begin{vmatrix}\mu & \nu\\ Q & R\end{vmatrix}}=\frac{\mu'}{\begin{vmatrix}\nu & \lambda\\ R & P\end{vmatrix}}=\frac{\nu'}{\begin{vmatrix}\lambda & \mu\\ P & Q\end{vmatrix}}\ \ldots\ldots\ldots\ldots\ldots(6);$$

and since the coordinates of the centre are

$$\frac{\bar{\alpha}}{\left(\frac{d\phi_1}{da}\right)}=\frac{\bar{\beta}}{\left(\frac{d\phi_1}{db}\right)}=\frac{\bar{\gamma}}{\left(\frac{d\phi_1}{dc}\right)}\ \ldots\ldots\ldots\ldots\ldots\ldots(7),$$

we have, from (6) and (7), for the equation of the diameter

$$\begin{vmatrix}\alpha & \beta & \gamma\\ \lambda' & \mu' & \nu'\\ \left(\frac{d\phi_1}{da}\right) & \left(\frac{d\phi_1}{db}\right) & \left(\frac{d\phi_1}{dc}\right)\end{vmatrix}=0.$$

2730. (Proposed by J. J. WALKER, M.A.)—If the determinant $D=\begin{vmatrix}a & e & f & g\\ e & b & h & i\\ f & h & c & k\\ g & i & k & d\end{vmatrix}$ vanish, prove that $(cd)=\dfrac{(ci)(i)+(dh)(h)}{(b)}$, where (b), (i), (h), (cd), ... are minors formed by omitting the rows and columns in which the bracketed constituents stand.

Solution by the PROPOSER.

In the *Messenger of Mathematics*, No. XIII., p. 27, I have shown that

$$\begin{vmatrix} (cd) & (dh) & (ci) \\ (dh) & (bd) & (bk) \\ (ci) & (bk) & (bc) \end{vmatrix} = a^2 D; \text{ consequently, when } D = 0,$$

$$(cd) = \frac{(bd)(ci)^2 + (bc)(dh)^2 - 2(dh)(ci)(bk)}{(bd)(bc) - (bk)^2}$$

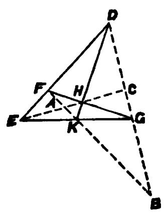
$$= \frac{(ci)\left\{(ci)(bd) - (bk)(dh)\right\} + (dh)\left\{(dh)(bc) - (bk)(ci)\right\}}{(bd)(bc) - (bk)^2}$$

$$= \frac{(ci)(i) + (dh)(h)}{(b)} \quad \text{(see p. 25)},$$

which is the type of six similar relations.

2808. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—Four fixed tangents are drawn to a conic S; three other conics are drawn osculating S in any the same point P, and each passing through the ends of a diagonal of the circumscribed quadrilateral; prove that the tangents to these conics at the ends of the diagonal meet in one point P′, and that the locus of P′ is a curve of the sixth degree and fourth class, having two cusps on every diagonal, and touching S at the points of contact of the four tangents.

Solution by F. D. THOMSON, M.A.



With the letters of the figure take the triangle ABC formed by the diagonals as the triangle of reference, and let the equation to the line DEF be $ax + by + cz = 0$. Then the coordinates of the points in the figure may be taken as

$$D(0, c, -b), \quad G(0, c, b),$$
$$E(-c, 0, a), \quad H(c, 0, a),$$
$$F(b, -a, 0), \quad K(b, a, 0).$$

By properly determining the constants, we can take as the equation to the given conic

$$x^2 + y^2 + z^2 = 0 \ldots\ldots\ldots\ldots \text{(i)}.$$

Since DEF touches this, we get the condition

$$a^2 + b^2 + c^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

But, by Salmon's *Conics*, Art. 251, the equation to a conic osculating (i) at the point $(x'y'z')$ is $x^2 + y^2 + z^2 + (xx' + yy' + zz')(lx + my + nz) = 0 \ldots\ldots$ (iii),

where $\quad lx' + my' + nz' = 0 \ldots\ldots\ldots\ldots\ldots\ldots$ (iv).

If (iii) pass through the points D and G, we find, substituting the values of the coordinates,

$$m = \frac{-y'(b^2 + c^2)}{(cy')^2 - (bz')^2} = \frac{a^2 y'}{(cy')^2 - (bz')^2}, \quad n = \frac{-a^2 z'}{(cy')^2 - (bz')^2},$$

and therefore $$l = \frac{a(z'^2 - y'^2)}{x'\{(cy')^2 - (bz')^2\}};$$

therefore (iii) becomes

$$S \equiv x^2 + y^2 + z^2 + \frac{a^2}{(cy')^2 - (bz')^2}\left\{(z'^2 - y'^2)\frac{x}{x'} + yy' - zz'\right\} \times \left\{xx' + yy' - zz'\right\} = 0.$$

Now, if (xyz) be the pole of DG, *i. e.* of $x = 0$, (xyz) is given by

$$\frac{dS}{dy} = 0, \quad \frac{dS}{dz} = 0.$$

These equations, after reductions by means of (ii) and $x'^2 + y'^2 + z'^2 = 0$, become

$$\frac{b^2 x'^2}{a^2} y - \frac{y'^3}{x'} x = 0 = \frac{-c^2 x'^2}{a^2} z + \frac{z'^3}{z} x,$$

or $$\frac{a^2 x}{x'^3} = \frac{b^2 y}{y'^3} = \frac{c^2 z}{z'^3} \quad \dots\dots\dots\dots\dots\dots \text{(v)},$$

which, from its symmetrical form, is evidently the same point as we should have obtained from either of the other two osculating conics which pass through EH and FK. Equations (v) give us

$$\frac{x'^2}{(a^2 x)^{\frac{2}{3}}} = \frac{y'^2}{(b^2 y)^{\frac{2}{3}}} = \frac{z'^2}{(c^2 z)^{\frac{2}{3}}} = \frac{0}{(a^2 x)^{\frac{2}{3}} + \text{\&c.}};$$

therefore the locus of the point is the curve

$$(ax)^{\frac{2}{3}} + (b^2 y)^{\frac{2}{3}} + (c^2 z)^{\frac{2}{3}} = 0, \quad \text{or} \quad (a^4 x^2 + b^4 y^2 + c^4 z^2)^3 = 17 a^4 b^4 c^4 x^2 y^2 z^2,$$

a curve of the sixth order having two cusps on each diagonal, at the points where they are cut by the conic

$$a^4 x^2 + b^4 y^2 + c^4 z^2 = 0.$$

If we transform to *tangential* coordinates, the equation to the point given by (v) is $$\frac{x'^3}{a^2}\lambda + \frac{y'^3}{b^2}\mu + \frac{z'^3}{c^2}\nu = 0 \quad \dots\dots\dots\dots \text{(vi)};$$

therefore, to find the locus, differentiate, and we have

$$\frac{x'^2}{a^2}\lambda\, dx' + \frac{y'^2}{b^2}\mu\, dy' + \frac{z'^2}{c^2}\nu\, dz' = 0, \quad \text{when} \quad x'dx' + y'dy' + z'dz' = 0;$$

therefore $$\frac{x'}{a^2}\lambda = \frac{y'}{b^2}\mu = \frac{z'}{c^2}\nu;$$

therefore, eliminating x', y', z', $$\frac{a^4}{\lambda^2} + \frac{b^4}{\mu^2} + \frac{c^4}{\nu^2} = 0 \quad \dots\dots\dots\dots\dots\dots \text{(vii)},$$

the tangential equation to the locus. This is of the 4th class, and is satisfied by the lines $(\pm a, \pm b, \pm c)$, and therefore touches the sides of the quadrilateral. Also the pole of the line $(\lambda'\mu'\nu')$ is $\lambda\lambda'(b^4\nu'^2 + c^4\mu'^2) + \text{\&c.} = 0$; therefore point of contact of (a, b, c), *i. e.* of DEF, is $a\lambda + b\mu + c\nu = 0$, or in trilinears $\frac{x}{a} + \frac{y}{b} = \frac{z}{c}$, which is the point of contact of DEF with the given conic; therefore, &c.

COR.—If the points F and K be the two real foci of the given conic, and E and H the circular points at infinity, B is the centre and DG the minor axis. Hence the osculating conic through EH and the point $(x'y'z')$ becomes the circle of curvature at $x'y'z'$; that through DG and $(x'y'z')$ an osculating conic through the imaginary foci; and that through FK and $(x'y'z')$ an oscu-

lating conic through the real foci. Hence the locus of the centre of curvature at $(x'y'z')$ is the same as that of the poles of the minor and major axes of the original conic with respect to the other two osculating conics.

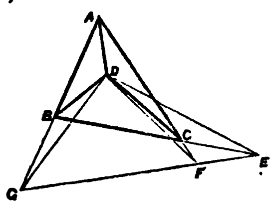
2787. (Proposed by C. W. Merrifield, F.R.S.) — The value of $\iiint x^2\,dx\,dy\,dz$ taken over the whole volume of a tetrahedron through one of whose corners the plane of (yz) passes, is $\frac{1}{5}V(a_1^2+b_1^2+c_1^2)$, where V is the volume, and a_1, b_1, c_1 are the distances of the middle points of the opposite edges from the plane of reference.

Solution by the Proposer.

Let us first find the value of the integral taken over the volume of a pyramid having its *base* in the plane of reference. Let the pyramid have a square base, and let one of its edges be perpendicular to its base. Let its height be h and the side of its base k. The Cartesian equation of one of its edges will be $\frac{x}{h}+\frac{y}{k}=1$, and we shall have to find $\int_0^h x^2y^2\,dx$ subject to that equation. On substituting $k\left(1-\frac{x}{h}\right)$ for y, and performing the integrations, we find its value to be $\frac{1}{30}k^2h^3=\frac{1}{30}$ base $\times$ (altitude)^3. It is evident that, whatever be the shape of the pyramid, we shall always have

$$\iiint x^2\,dx\,dy\,dz=\tfrac{1}{30}\text{ base}\times(\text{altitude})^3,$$

provided the base of the pyramid be in the plane of reference. Moreover, the volume is $\frac{1}{3}$ base $\times$ (altitude).



Now let ABCD be the tetrahedron, and DEFG the plane of reference. Produce the four planes of the pyramid to intersect with DEFG, as shown in the figure, and let the distances of A, B, and C from the plane ot reference be α, β, and γ respectively.

Since the pyramids ADFG and BDEF intersect so as to have the quadrangular pyramid DBCFG for a common frustum, we have

$$\text{ABCD}=\text{ADFG}+\text{CDEF}-\text{BDEG};$$

and also, if we call MI the integral $\iiint x^2\,dx\,dy\,dz$ taken over the corresponding pyramid, we have

$$\text{MI}\,.\,\text{ABCD}=\text{MI}\,.\,\text{ADFG}+\text{MI}\,.\,\text{CDEF}-\text{MI}\,.\,\text{BDEG}.$$

But $$\text{MI}\,.\,\text{ADFG}=\frac{\alpha^3}{30}\text{DFG}=\frac{\alpha^2}{10}\text{ADFG},$$

and similarly $$\text{MI}\,.\,\text{CDEF}=\frac{\gamma^2}{10}\text{CDEF},\quad \text{MI}\,.\,\text{BDEG}=\frac{\beta^2}{10}\text{BDEG}.$$

Again, $$\text{BDEG}=\text{ADFG}+\text{CDEF}-\text{ABCD};$$

therefore $$10\text{MI}\,.\,\text{ABCD}=(\alpha^2-\beta^2)\,\text{ADFG}+(\gamma^2-\beta^2)\,\text{CDEF}+\beta^2\,.\,\text{ABCD};$$

next, $\quad \text{AFGD} : \text{ABCD} = \triangle\text{AFG} : \triangle\text{ABC} = \text{AF}.\text{AG} : \text{AC}.\text{AB}$
$$= \alpha.\alpha : (\alpha-\gamma)(\alpha-\beta);$$
also, $\quad \text{CEFD} : \text{ABCD} = \triangle\text{CEF} : \triangle\text{ABC} = \text{FC}.\text{CE} : \text{AC}.\text{CB}$
$$= \gamma.\gamma : (\alpha-\gamma)(\beta-\gamma);$$

$$\therefore \frac{10\,\text{MI}.\text{ABCD}}{\text{vol. ABCD}} = \frac{\alpha^2(\alpha^2-\beta^2)}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\gamma^2(\gamma^2-\beta^2)}{(\gamma-\alpha)(\gamma-\beta)} + \beta^2$$
$$= \frac{\alpha^2(\alpha+\beta)}{\alpha-\gamma} - \frac{\gamma^2(\gamma+\beta)}{\alpha-\gamma} + \beta^2 = \frac{\alpha^3-\gamma^3+\beta(\alpha^2-\gamma^2)}{\alpha-\gamma} + \beta^2$$
$$= \alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta$$
$$= \tfrac{1}{2}(\alpha+\beta)^2 + \tfrac{1}{2}(\beta+\gamma)^2 + \tfrac{1}{2}(\gamma+\alpha)^2;$$

therefore $\quad \text{MI}.\text{ABCD} = \tfrac{1}{5}\text{V}(a_1^2+b_1^2+c_1^2).$

MI is not a moment of inertia, being taken with reference to a plane instead of a line; but the formula, as it involves the squares, evidently holds for the moment of inertia about an axis. The common moment about the plane is

$$\tfrac{1}{4}\text{V}(a_1+b_1+c_1).$$

Moreover, the common moment and moment of inertia of a triangle about a line passing through one of its angular points are, severally, A being the area of the triangle,

$$\tfrac{1}{3}\text{A}(a_1+b_1+c_1) \quad \text{and} \quad \tfrac{1}{6}\text{A}(a_1^2+b_1^2+c_1^2).$$

The similarity of these forms is very remarkable.

2701. (Proposed by the Rev. J. Wolstenholme, M.A.)—In a right conoid whose axis is the axis of z, show that the radii of principal curvature at the point $(r\cos\theta,\ r\sin\theta,\ z)$ are given by the equation

$$4\rho^2\left(\frac{dz}{d\theta}\right)^2 + r\rho\frac{d^2z}{d\theta^2}\left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^{\frac{1}{2}} = \left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^2.$$

I. *Solution by the* Rev. J. L. Kitchin, M.A.

The radii of principal curvature are given by the equation

$$(r_1t-s^2)\rho^2 - \left\{(1+p^2)t - 2pqs + (1+q^2)r_1\right\}\rho(1+p^2+q^2)^{\frac{1}{2}}$$
$$+ (1+p^2+q^2)^2 = 0 \ldots\ldots (1).$$

The general equation of a conoid, axis of surface axis of z, is $z = \phi\left(\dfrac{y}{x}\right)$;

and since $x = r\cos\theta$, $y = r\sin\theta$, this becomes $z = \phi(\tan\theta) = f(\theta)$; therefore z is independent of r. Hence we find

$$\frac{dz}{dx} = -\frac{1}{r}\sin\theta\frac{dz}{d\theta},\quad \frac{dz}{dy} = \frac{1}{r}\cos\theta\frac{dz}{d\theta};$$

and hence $\quad 1+p^2+q^2 = 1+\dfrac{1}{r^2}\left(\dfrac{dz}{d\theta}\right)^2 = \dfrac{1}{r^2}\left\{r^2+\left(\dfrac{dz}{d\theta}\right)^2\right\}; \quad$ also

$$\frac{d^2z}{dx^2} = -\frac{d}{dx}\left(\frac{1}{r}\sin\theta\frac{dz}{d\theta}\right) = -\frac{d}{dr}\left(\frac{1}{r}\sin\theta\frac{dz}{d\theta}\right)\frac{dr}{dx} - \frac{d}{d\theta}\left(\frac{1}{r}\sin\theta\frac{dz}{d\theta}\right)\frac{d\theta}{dx}.$$

Effecting the differentiations and reducing, remembering that $\dfrac{dz}{d\theta}$ is a function of θ *only*, we get

$$\frac{d^2z}{dx^2} = \frac{\sin^2\theta}{r^2}\frac{d^2z}{d\theta^2} + \frac{2\sin 2\theta}{r^2}\frac{dz}{d\theta} = r_1;$$

similarly $$\frac{d^2z}{dy^2} = \frac{\cos^2\theta}{r^2}\frac{d^2z}{d\theta^2} - \frac{2\sin 2\theta}{r^2}\frac{dz}{d\theta} = t;$$

and $$\frac{d^2z}{dx\,dy} = -\left(\frac{2\cos 2\theta}{r^2}\frac{dz}{d\theta} + \frac{\sin 2\theta}{2r^2}\frac{d^2z}{d\theta^2}\right);$$ hence

$$r_1t - s^2 = \frac{1}{r^4}\left\{\frac{\sin^2 2\theta}{4}\frac{d^2z}{d\theta^2} + \sin 4\theta\frac{d^2z}{d\theta^2}\frac{dz}{d\theta} - 4\sin^2 2\theta\left(\frac{dz}{d\theta}\right)^2\right\}$$

$$-\frac{1}{r^4}\left\{4\cos^2 2\theta\left(\frac{dz}{d\theta}\right)^2 + \sin 4\theta\frac{d^2z}{d\theta^2}\frac{dz}{d\theta} + \frac{1}{4}\cos^2 2\theta\left(\frac{d^2z}{d\theta^2}\right)^2\right\}$$

$$= -\frac{4}{r^4}\left(\frac{dz}{d\theta}\right)^2 \ldots\ldots\ldots$$

We find also that $(1+p^2)\,t - 2pqs + (1+q^2)\,r_1 = \dfrac{1}{r^2}\dfrac{d^2z}{d\theta^2}$.

Hence (1) becomes

$$-\frac{4}{r^4}\left(\frac{dz}{d\theta}\right)^2\rho^2 - \frac{1}{r^2}\rho\frac{d^2z}{d\theta^2}\frac{\left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^{\frac{1}{2}}}{r} + \frac{\left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^2}{r^4} = 0,$$

or $$4\rho^2\left(\frac{dz}{d\theta}\right) + r\rho\frac{d^2z}{d\theta^2}\left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^{\frac{1}{2}} = \left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^2.$$

II. *Solution by the* PROPOSER.

Let the increments of r, θ, z, at a point Q near the given point P, be $\lambda\delta\theta$, $\delta\theta$, δz; then the tangent plane at P is the limiting position of the plane containing the generating line $X\sin\theta = Y\cos\theta$, $Z=z$, and passing through an adjacent point of the surface; its equation being taken $X\sin\theta - Y\cos\theta + k(Z-z) = 0$, k is the limit of $\dfrac{(r+\lambda)\sin\delta\theta}{\delta z}$, or $r \div \dfrac{dz}{d\theta}$. But if ρ be the radius of curvature of the normal section containing the ultimate position of the line PQ, and p be the perpendicular from Q on the tangent plane at P, $2\rho =$ limit of $\dfrac{PQ^2}{p}$.

Now $$PQ^2 = \lambda^2\delta\theta^2 + r^2\delta\theta^2 + \delta z^2,$$

and $$p = \frac{-(r+\lambda\delta\theta)\frac{dz}{d\theta}\sin\delta\theta + r\left(\frac{dz}{d\theta}\delta\theta + \frac{d^2z}{d\theta^2}\frac{\delta\theta^2}{z} + \ldots\right)}{\left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^{\frac{1}{2}}}$$

whence $$\rho = \left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^{\frac{1}{2}}\frac{r^2+\left(\frac{dz}{d\theta}\right)^2+\lambda^2}{r\frac{d^2z}{d\theta^2} - \lambda\frac{dz}{d\theta}};$$

and the principal radii of curvature are the maximum and minimum values of this for different values of λ, and are therefore given by the equation

$$4\rho^2\left(\frac{dz}{d\theta}\right)^2 + r\rho\frac{d^2z}{d\theta^2}\left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^{\frac{1}{2}} = \left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^2.$$

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